The Natural Transform Decomposition Method For Solving Fractional Differential Equations

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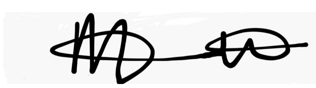
"The Natural Transform Decomposition Method For Solving Fractional Differential Equations"

I declare that the above dissertation is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.



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Abstract

In this dissertation, we use the Natural transform decomposition method to obtain approximate analytical solution of fractional differential equations. This technique is a combination of decomposition methods and natural transform method. We use the Adomian decomposition, the homotopy perturbation and the Daftardar-Jafari methods as our decomposition methods. The fractional derivatives are considered in the Caputo and Caputo-Fabrizio sense.

Keywords: Fractional differential equations, special functions, natural transform, decomposition methods, Caputo fractional derivative, Caputo-Fabrizio fractional derivative, Adomian decomposition method, Homotopy perturbation method, Daftardar-Jafari method, integral transform.

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This dissertation is dedicated

to

My parents and my wife, without whom none of this would have been even possible. It is also dedicated to those who speak truth to power and make it their business to do so. In remembrance of

♡

Papers

(1) H. Jafari and M.N. Ncube, Fourier-Natural transform method for solving a class of fractional partial differential equations, Far East Journal of Mathematical Sciences (FJMS) 109(2) (2018) 419-428.

(2) H. Jafari, M.N. Ncube, S.P. Moshokoa, Natural Daftardar-Jafari Method for solving fractional partial differential equations, Nonlinear Dynamics and Systems Theory, 2018[Accepted].

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Chapter 1

Introductory Remarks

1.1 Objectives of the research

This research seeks to accomplish the following:

1) Apply the Natural transform decomposition method to fractional differential equations.

2) Compare the results from the Natural transform decomposition method with other methods that are already in existence for solving fractional differential equations.

3) Investigate the complexity and efficiency of the Natural transform decomposition method as compared to other methods for solving fractional differential equations.

4) Compare the results from the Caputo and Caputo-Fabrizio fractional derivatives.

1.2 Outline of the research

Chapter 1: This chapter focuses on the introduction of the research, literature review and some important definitions and notations of symbols that will be used in this research.

Chapter 2: This chapter introduces the aspects that form the basis of Fractional differential equations(FDEs), the chapter will concentrate on the definition and history of the different types of fractional derivatives and their associated integrals. We introduce this chapter by discussing special functions.

Chapter 3: This chapter focuses on the integral transform methods, we give definitions of the different types of integral transforms. Then we give more details of the natural transform, it's properties and illustrate it's use by giving examples.

Chapter 4: In this chapter we focus on the decomposition methods for solving fractional differential equations. We give a general description of three decomposition methods and thereafter we illustrate the use of these methods with an example and do convergence analysis.

Chapter 5: This chapter focuses on combining the natural transform and decomposition methods to come up with one method, the natural transform decomposition method. We will give a general description of the natural transform decomposition method as it applies to FDEs. We will then apply this method to the non linear Fractional Klein-Gordon differential equation, compare our results with other methods and do convergence analysis. In this chapter we will use the Caputo fractional derivative.

Chapter 6: This chapter will be like the previous one, chapter 5, the only difference is that our fractional derivative will be the Caputo-Fabrizio. We will make a comparison of the results from chapter 5 and 6.

Chapter 7: This chapter will summarize all the findings of the dissertation and state the areas that are still open for research.

1.3 Introduction and Literature review

Fractional Differential Equations (FDEs) are differential equations based on the subject of fractional calculus [36]. Fractional calculus is a branch of mathematics that deals with derivatives and integrals of arbitrary order, simply put, fractional differential equations allow the order of a derivative to be a fraction. That's integer order derivatives and integrals are embodied in fractional calculus [36].

The birth of fractional derivatives and integrals (fractional calculus) dates back from the year 1695. It was initiated by L Hopital, who asked Leibniz a question, what if the order of the derivative is $\frac{1}{2}$ [36]. Leibniz's response was, this will lead to a paradox from which

important results will be deduced [36].
For a long time since they were discovered, fractional derivatives were reserved for the mathematicians due to the fact that there were presented in the form of pure theory [36]. However, for the past two decades, many researchers have indicated that fractional derivatives are also suitable for modelling problems that arise in various fields such as science, engineering and medicine [1, 34, 36]. They have been used with success to model problems in chaos, fractals, random walks, systems that involve diffusion, dynamical systems and mechanics [1, 34, 36].

Since it has become apparent that FDEs have shown importance in various fields, it is equally important that there should be convenient, easy to implement and accurate methods to solve them. Despite noticeable progress in the field of FDEs, it is important to know that there are no universally agreed upon methods to solve them, thus there is still a lot that needs to be done concerning the methods of solving FDEs. Some of the methods that have been used include, perturbation techniques, variational iterative method, decomposition methods (iterative methods), numerical methods and integral transform methods [13, 21, 26, 28, 36].

Then fairly recently, researchers have considered the possibility of combining the integral transform methods and decomposition methods to come up with what we may call integral transform decomposition methods. This method is a two stage process, the first step is to transform the differential equation to be solved using an integral transform, then the second step is to apply the decomposition methods to the transformed differential equation. Some of the integral transforms that have been combined with the decomposition methods are the Laplace and the natural transforms [1, 20, 22, 25, 29, 31, 33–35, 37].

The Laplace transform was combined with the Adomian decomposition method to come up with the Laplace decomposition method (LDM), this method was used to solve linear and non linear wave and diffusion equations of fractional order [22]. Then again the Laplace transform was combined with the homotopy perturbation method to get the Laplace Homotopy perturbation method (LHPM), this method was used to solve the fractional Burgers equation [25].

The natural transform was combined with the Adomian decomposition method and the authors called this method the Adomian decomposition natural transform method or the Natural decomposition method (NDM) [1], this method was used to solve non linear fractional partial differential equations. The homotopy perturbation method was combined with the natural transform to come up with the Natural homotopy perturbation method (NHPM) [33], this method was applied to the linear and non linear fractional differential equations.

It is of importance to know that the integral transform decomposition methods are not only used to solve FDEs, but can also be successfully applied to integer order differential equations. The LDM was applied with success to solve linear and nonlinear ordinary differential equations in [29] and initial value problems in [31]. The NDM was used successfully to solve coupled systems of nonlinear partial differential equations in [37].

In this dissertation we focus on combining the natural transform with the Adomian decomposition and homotopy perturbation methods. In addition to this we will propose our own method by combining the natural transform and an iterative (decomposition) method known as the Daftardar-Jafari method.

1.4 Important notations

In table 1.1 we define important symbols and notations that we will frequently use throughout the dissertation.

Table 1.1: Table showing important symbols and notations.

Symbol	Meaning
$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Real numbers
$\mathbb{R}^n$	n dimensional Euclidean space
$\mathbb{C}$	Set of complex numbers
H	Hilbert space
X	Banach space
$\mathfrak{D}^\mu$	The Riemann-Liouville fractional derivative of order μ
$\mathcal{D}^\mu$	The Caputo fractional derivative of order μ
$\mathbb{D}^\mu$	The Caputo-Fabrizio fractional derivative of order μ

Chapter 2

Fractional Calculus

2.1 Introduction

In this chapter we will deal with the basic theory of fractional calculus. We will first start by discussing special functions and then move on to fractional calculus. There are numerous different approaches that have been put forward by researchers to define the fractional derivative, more detail on this can be found in [36]. In this dissertation we are going to limit ourselves to the Caputo and Caputo Fabrizio fractional derivatives.

However when it comes to the choice of terminals, there are only two options available, the left and the right. A fractional derivative of $y(t)$ with respect to t of order μ given as ${}_aD_t^{\mu}y(t)$ with lower terminal a fixed and upper terminal t allowed to move is called a left fractional derivative [36]. On the other hand, a fractional derivative of $y(t)$ with respect to t of order μ given as ${}_tD_b^{\mu}y(t)$ with lower terminal t allowed to move and upper terminal b fixed is known as the right fractional derivative [36].

In the practical sense, the left fractional derivative explains how the present state of $y(t)$ is dependent upon its historical states, and the right fractional derivative explains how the present state of $y(t)$ depends on it's future states [36]. Since from a practical viewpoint, we do not have an idea how the current state of a process is affected by its future states, then for this dissertation we are going to restrict ourselves to the left fractional derivatives. In addition to this, our lower terminal is going to be fixed at 0, that is ($a = 0$).

It is important to know that from the mathematical theory, in particular, the theory concerning boundary value problems for FDEs can be developed only by using both the left and right fractional derivatives [36].

2.2 Special functions

Special functions play a crucial role in the development of the theory of fractional derivatives [36]. In this dissertation we are going to limit ourselves to the Euler's Gamma and Mittag-Leffler functions, for a detailed discussion on other special functions see [36].

The Euler's gamma function is used to generalize the factorial ($n!$), it also accommodates n to be non-integers and complex numbers [36]. We give the definition of the Euler's gamma function below.

Definition 2.2.1 *The Gamma function,* $\Gamma(z)$ *is represented by the following integral [36],*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad t > 0. \tag{2.1}$$

The Gamma function has the following two[1] important properties [36]

(i) $\Gamma(z+1) = z\Gamma(z)$.

(ii) $\Gamma(n+1) = n!, \quad n \in \mathbb{N}$.

We prove the second property, using (2.1)

$$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt = \int_0^\infty e^{-t} dt = [-e^{-t}]_0^\infty = 1.$$

[1] See [36] where the third property of the Gamma function is also discussed.

Then using the first property, we get

$$\Gamma(2) = 1 \times \Gamma(1) = 1 \times 1! = 1!,$$
$$\Gamma(3) = 2 \times \Gamma(2) = 2 \times 1! = 2!,$$
$$\Gamma(4) = 3 \times \Gamma(3) = 3 \times 2! = 3!.$$

Thus, we can deduce that

$$\Gamma(n+1) = n \times \Gamma(n) = n \times (n-1)! = n!.$$

The Mittag-Leffler function is a general function with common functions like the exponential, sine and cosine being some of it's special cases [36].

Definition 2.2.2 *The Mittag-Leffler function in one parameter is defined as [36]*

$$E_{\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + 1)}, \qquad z \in \mathbb{C}, \quad \mu > 0. \tag{2.2}$$

Definition 2.2.3 *The Mittag-Leffler function in two parameters is defined as [36]*

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\mu k + \nu)}, \qquad z \in \mathbb{C}, \quad \mu, \nu > 0. \tag{2.3}$$

We mention how the Mittag-Leffler function is related to some of it's special cases below.

The exponential function is a special case of the Mittag-Leffler function if [36],

$$E_{1,1}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)} = e^t, \quad t > 0. \tag{2.4}$$

The cosine function is a special case of the Mittag-Leffler function if [36]

$$E_{2,1}(-c^2t^2) = \sum_{k=0}^{\infty} \frac{(-c^2t^2)^k}{\Gamma(2k+1)} = \cos(ct), \quad c \in \mathbb{R}, \quad t > 0. \tag{2.5}$$

The sine function is a special case of the Mittag-Leffler function if [36]

$$E_{2,2}(-c^2t^2) = t \sum_{k=0}^{\infty} \frac{(-c^2t^2)^k}{\Gamma(2k+2)} = \frac{1}{ct} \sin(ct), \quad c \in \mathbb{R}, \quad c \neq 0, \quad t > 0. \tag{2.6}$$

There are other types of special functions that we didn't mention here like the Beta and the Wright functions, more detail is given on the special functions in [36]. It is also discussed in [36] how the Beta and Wright functions are related to the Mittag-Leffler function.

2.3 Fractional derivatives

We have already mentioned that there are numerous definitions of the fractional derivative that are at our disposal, here we are going to briefly discuss a few of them and give their definitions. These will be the Riemann-Liouville, the Caputo, the Caputo-Fabrizio and the Atangana-Baleanu-Liouville fractional derivatives.

We will start the discussions of fractional derivatives by mentioning the Riemann-Liouville's fractional integral.

Definition 2.3.1 *The Riemann-Liouville fractional integral of order μ of the function $y(t)$ is given as [36]*

$$I_t^{\mu} y(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \int\limits_0^t \frac{y(s)ds}{(t-s)^{1-\mu}}, & \text{if} \quad \mu > 0, \quad t > 0, \quad ; \\ y(t), & \text{if} \quad \mu = 0. \end{cases} \tag{2.7}$$

Γ *is the Euler's Gamma function.*

We note the following properties of the Riemann-Liouville's fractional integral [21],

(i) $I_t^{\mu} I_t^{\alpha} y(t) = I_t^{\mu+\alpha} y(t), \quad \mu, \alpha > 0.$

(ii) $I_t^{\mu} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)} t^{\mu+\nu}, \quad 0 < \mu, \quad -1 < \nu, \quad t > 0.$

(iii) $I_t^{\mu} \mathfrak{D}_t^{\mu} y(t) = y(t) - \sum\limits_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0), \quad \mu \in (p-1, p],$
$\mathfrak{D}_t^{\mu}$ is the Riemann-Liouville fractional derivative that we will shortly define.

The Riemann-Liouville fractional integral is also used in the Caputo fractional derivative [36].

We give the definitions of the fractional derivatives below, starting with the Riemann-Liouville's fractional derivative.

Definition 2.3.2 *The Riemann-Liouville fractional derivative of order μ of the function*

$y(t)$ for $p \in \mathbb{N}$ is stated as [36],

$$\mathfrak{D}_t^{\mu} y(t) = \begin{cases} \frac{d^p}{dt^p}\left[\frac{1}{\Gamma(p-\mu)}\int\limits_0^t \frac{y(s)ds}{(t-s)^{\mu-p+1}}\right] & \textit{if} \quad p-1<\mu<p; \quad t>0 \\ \frac{d^p}{dt^p} y(t) & \textit{if} \quad \mu = p. \end{cases} \tag{2.8}$$

The Riemann-Liouville fractional derivative played a pivotal role in the development of fractional derivatives, mostly providing the basis for the study of FDEs [36].

However, even though problems involving the Riemann-Liouville's fractional derivative can be solved with great success in mathematics, unfortunately these solutions lack practical meaning in applied problems [36]. This is simply because there are no known physical interpretations for their fractional order initial conditions [36].

The solution to this problem was tabled by Michele Caputo. Caputo's definition of the fractional derivative allow the initial conditions of the FDEs to be the same as that of the integer order differential equations [36].

Definition 2.3.3 *The Caputo fractional derivative of order μ of the function $y(t)$ for $p \in \mathbb{N}$ is stated as [36],*

$$\mathcal{D}_t^{\mu} y(t) = \begin{cases} \frac{1}{\Gamma(p-\mu)}\int\limits^t \frac{y^{(p)}(s)ds}{(t-s)^{\mu-p+1}} & \textit{if} \quad p-1<\mu\leq p; \\ \frac{d^p}{dt^p} y(t) & \textit{if} \quad \mu = p. \end{cases} \tag{2.9}$$

If μ is a natural number in (2.9), then the Caputo definition of the fractional derivative coincides with the integer order derivatives.

In 2015, Michele Caputo and Mauro Fabrizio modified the Caputo fractional derivative to come up with a new derivative, the Caputo-Fabrizio fractional derivative [12]. This new fractional derivative was obtained by substituting the kernel in the Caputo fractional derivative with an exponential function to get the fractional derivative without singular kernel [12].

Definition 2.3.4 *The Caputo-Fabrizio fractional derivative of order α of the function $y(t)$*

is stated as [12],

$$\mathbb{D}_t^{\alpha} y(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t y'(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] ds, \qquad 0 < \alpha \leq 1. \tag{2.10}$$

$M(\alpha)$ is the normalisation function that satisfies $M(0) = M(1) = 1$. It is observed that at $t = s$, the kernel $(t-s)^{-\mu+p-1}$ of the Caputo has a singularity, simply put, it's kernel is not defined at this point. However for the Caputo-Fabrizio, the kernel $\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]$ does not have a singularity at $t = s$. We also observe that $\frac{1}{\Gamma(p-\mu)}$ in the Caputo fractional derivative is replaced with $\frac{M(\alpha)}{(1-\alpha)}$. The similarity in (2.9) and (2.10) is that both fractional derivatives will be zero if $y(t)$ is a constant.

The Caputo-Fabrizio fractional derivative was then modified by A. Atangana and D. Baleanu to come up with the Atangana-Baleanu-Liouville fractional derivative [40].

Definition 2.3.5 *The Atangana-Baleanu fractional derivative of order μ of the function $y(t)$ for $y \in H^1(a,b) \quad a < b$ is stated as [40],*

$$\mathbf{D}_t^{\mu} y(t) = \frac{M(\mu)}{1-\mu} \int_a^t y'(s) E_{\mu}\left[-\frac{\mu(t-s)}{1-\mu}\right] ds \qquad 0 \leq \mu \leq 1. \tag{2.11}$$

$M(\mu)$ has similar properties as in the Caputo-Fabrizio fractional derivative and E_{μ} is the Mittag-Leffler function that we gave in (2.2). It is observed that the Atangana-Baleanu-Liouville fractional derivative has the Mittag-Leffler function as the kernel in place of the exponential function in the Caputo-Fabrizio fractional derivative.

We will not discuss the fractional integrals of the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives in this dissertation, a discussion on these can be found in [40].

Chapter 3

Integral Transform Methods

3.1 Introduction

The integral transform methods are based on the integral transforms. Integral transforms act as operators or mappings that transform the original domain of the original problem into another domain. To get the solution of the original problem, an inverse integral transform is used to transform the new domain back to the original domain. Integral transform methods tend to be particularly useful in instances when the original problem is difficult to solve, and transforming the problem often lessens it's difficulty.

There are various types of integral transform methods that are used to solve linear differential equations. These include, the Laplace transform method (LTM) [17, 26, 36], the natural transform method(NTM) [9, 28], the Sumudu transform method (STM) [8], the Elzaki transform method (ETM) [41], the Fourier transform method (FTM) [17] and the Mellin transform method (MTM) [36]. Among all of the integral transform methods that have been used up to date, the LTM is the most popular [28]. It has become a tradition that every new integral transform that is discovered is investigated how it links with the Laplace transform.

In this chapter our main focus will be on the natural transform, we give it's definition, establish it's connection with the Laplace transform, state and prove some of it's properties, give a short table of some properties and formulae with the Laplace transform included. We will prove the natural transforms of, the Caputo fractional derivative, Caputo Fabrizio fractional derivative and the Mittag-Leffler function. We will also give examples to illustrate the use of the NTM.

We will start by giving the definition of the Laplace transform before we move on to the natural transform.

Definition 3.1.1 *The Laplace transform of the function $y(t)$ for $a \in \mathbb{R}$ and some positive constants K and T is defined in the set,*
$S=\{y(t) :| y(t) |\leq Ke^{at}, \quad if \quad t > T\}$,
by the integral

$$\mathcal{L}[y(t)] = Y(s) = \int_0^\infty e^{-st} y(t)dt, \quad s > 0 \tag{3.1}$$

[17, 26, 36].

In the above definition, s is the Laplace transform complex variable.

To get back the original function from $Y(s)$ in the definition above we apply the inverse Laplace transform defined as [17],

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \int_{c-i\infty}^{c+i\infty} e^{st} Y(s)ds, \quad c \in \mathbb{R}. \tag{3.2}$$

To evaluate the inverse Laplace transform using (3.2) is often difficult and a table is usually used as an alternative, we will give this table later in this chapter.

3.2 Natural transform and it's properties

The natural transform method (NTM) is one of the integral transform methods, and is based on the natural transform. It is a relatively new and powerful technique for solving linear differential equations. This method was introduced by Z. H Khan and W. Khan, they demonstrated the use of this method by solving an unsteady viscous flow problem in [28]. In [9], the NTM was applied to the Maxwell's equations to get the transient solutions of the magnetic and electric fields. The NTM was also used to solve fractional linear partial differential equations in [23].

We give the definition of the natural transform below.

Definition 3.2.1 *The natural transform of the function $y(t)$ for some positive constant K is defined in the set,*
S=$\left\{ y(t) : \tau_1, \tau_2 > 0, | y(t) | \leq K e^{\frac{|t|}{\tau_j}}, \quad if \quad t \in (-1)^j \times [0, \infty), j = 1, 2, ... \right\}$,
by the integral

$$\mathcal{N}[y(t)] = \psi(s, u) = \int_0^\infty e^{-st} y(ut) dt, \qquad t, s, u > 0, \tag{3.3}$$

[9, 28].

In the above definition, s and u are the natural transform variables.

To get back the original function from $\psi(s, u)$ in (3.3) we use the inverse natural transform defined as [28],

$$\mathcal{N}^{-1}[\psi(s, u)] = y(t) = \int_{c-i\infty}^{c+i\infty} e^{st} \psi(s, u) ds, \quad c \in \mathbb{R}. \tag{3.4}$$

The same difficulty is also encountered in evaluating (3.4) as in (3.2), thus information concerning the natural transforms of known functions is often used, we give this information later in table 3.1.

The natural transform has the following properties [9, 28].

(i) It is a linear operator. Given functions $y_1(t)$ and $y_2(t)$ with defined natural transforms and constants k_1,$k_2 \in \mathbb{R}$, then

$$\mathcal{N}[k_1 y_1(t) + k_2 y_2(t)] = k_1 \mathcal{N}[y_1(t)] + k_2 \mathcal{N}[y_2(t)].$$

(ii) It exhibits time scaling property,

$$\mathcal{N}[y(ct)] = \frac{1}{c} \psi\left(\frac{s}{c}, u\right) \qquad t > 0, \quad c \in \mathbb{R}, \quad c \neq 0.$$

(iii)
$$\mathcal{N}[y'(t)] = \frac{s}{u}\psi(s,u) - \frac{1}{s}y(0).$$

Proof using, (3.3)

$$\mathcal{N}[y'(t)] = \int_0^\infty e^{-st}y'(ut)dt = \left[\frac{1}{u}e^{-st}y(ut)\right]_0^\infty + \frac{s}{u}\int_0^\infty e^{-st}y(ut)dt = \frac{s}{u}\psi(s,u) - \frac{1}{u}y(0).$$

(iv)
$$\mathcal{N}[y''(t)] = \left(\frac{s}{u}\right)^2 \psi(s,u) - \frac{s}{u^2}y(0) - \frac{1}{u}y'(0).$$

Proof using, (3.3)

$$\mathcal{N}[y''(t)] = \int_0^\infty e^{-st}y''(ut)dt = \left[\frac{1}{u}e^{-st}y'(ut)\right]_0^\infty + \frac{s}{u}\int_0^\infty e^{-st}y'(ut)dt = \left(\frac{s}{u}\right)^2 \psi(s,u) - \frac{s}{u^2}y(0) - \frac{1}{u}y'(0).$$

To generalise to higher order derivatives, the concept of mathematical induction is used.

We will demonstrate how to calculate the natural transforms of the basic functions using the exponential function.
If $y(t) = e^{ct}$ where c is a constant and $t > 0$, the natural transform of $y(t)$ is given by

$$\mathcal{N}[y(t)] = \int_0^\infty e^{-st}e^{uct}dt = \int_0^\infty e^{(-s+uc)t}dt = \frac{1}{s-cu}. \tag{3.5}$$

The natural transforms of other basic functions are given in table 3.1.

We state and prove a theorem that links the natural and Laplace transforms below.

Theorem 3.2.1 *Given a function $y(t)$ with it's natural and Laplace transforms $\psi(s,u)$ and $Y(s)$ respectively then,*

$$\psi(s,u) = \frac{1}{u}Y\left(\frac{s}{u}\right). \tag{3.6}$$

Proof

Let $\beta = ut$, and we substitute in (3.3),

$$\mathcal{N}[y(t)] = \psi(s,u) = \frac{1}{u}\int_0^\infty e^{-\frac{s}{u}\beta}y(\beta)d\beta = \frac{1}{u}Y\left(\frac{s}{u}\right).$$

We observe from the above theorem that if $u = 1$, the natural transform becomes the Laplace transform, when we compare the definitions of the two transforms (3.3) and (3.1) this is indeed true.

In table 3.1 we provide the natural and Laplace transforms of some basic functions and fundamental properties.

Table 3.1: Table showing properties and formulae for natural and Laplace transforms.

Category	y(t)	$\psi(s,u)$	$Y(s)$
Basic functions	1	$\frac{1}{s}$	$\frac{1}{s}$
	$sin\omega t$	$\frac{u\omega}{s^2+u^2\omega^2}$	$\frac{\omega}{s^2+\omega^2}$
	$cos\omega t$	$\frac{s}{s^2+u^2\omega^2}$	$\frac{s}{s^2+\omega^2}$
	e^{ct}	$\frac{1}{s-cu}$	$\frac{1}{s-c}$
	$\sinh ct$	$\frac{cu}{s^2-c^2u^2}$	$\frac{c}{s^2-c^2}$
	$\cosh ct$	$\frac{s}{s^2-c^2u^2}$	$\frac{s}{s^2-c^2}$
	$\frac{1}{\Gamma(n)}t^{n-1}$	$u^{n-1}s^{-n}$	s^{-n}
Fundamental properties	$e^{ct}y(t)$	$\frac{s}{s-cu}(\frac{su}{s-cu})$	$Y(s-c)$
	$y'(t)$	$\frac{s}{u}\psi(s,u) - \frac{1}{u}y(0)$	$sY(s) - y(0)$
	$y''(t)$	$(\frac{s}{u})^2\psi(s,u) - \frac{s}{u^2}y(0) - \frac{1}{u}y'(0)$	$s^2Y(s) - sy(0) - y'(0)$
	$\underbrace{H(t-d)y(t-d)}_{H \text{ is a Heaviside function}}$	$e^{-d\frac{s}{u}}\psi(s,u)$	$e^{-ds}Y(s) \quad \forall \quad p > 0$
Convolution	$\int\limits_0^t y_1(t-k)y_2(k)dk$	$u\psi_1(s,u)\psi_2(s,u)$	$Y_1(s)Y_2(s)$

In the next two examples, we demonstrate how to use the natural transform method to solve ordinary differential equations with constant coefficients.

Example 3.1 *Consider the following ordinary differential equation given in [17],*

$$y'(t) + y(t) = 1, \tag{3.7}$$
$$y(0) = 2.$$

We take the natural transform on both sides of (3.7)

$$\mathcal{N}[y'(t)] + \mathcal{N}[y(t)] = \mathcal{N}[1], \tag{3.8}$$

using the fundamental properties of the natural transform from table 3.1 the above equation gives,

$$\frac{s}{u}\psi(s,u) - \frac{1}{u}y(0) + \psi(s,u) = \frac{1}{s}.$$

We then substitute for the initial condition and solve for $\psi(s,u)$ to get

$$\psi(s,u) = \frac{u}{s(s+u)} + \frac{2}{s+u} = \frac{1}{s} + \frac{1}{s+u}. \tag{3.9}$$

Taking the inverse natural transform on both sides of (3.9) using table 3.1 yields

$$y(t) = 1 + e^{-t}. \tag{3.10}$$

This is the same solution that was obtained in [17] using the Laplace transform method.

Example 3.2 *Consider the following ordinary differential equation given in [17],*

$$y''(t) + 4y(t) = 3, \tag{3.11}$$
$$y(0) = 1, \quad y'(0) = 5.$$

Taking the natural transform on both sides of (3.11) leads us to

$$\left(\frac{s}{u}\right)^2 \psi(s,u) - \frac{s}{u^2}y(0) - \frac{1}{u}y'(0) + 4\psi(s,u) = \frac{3}{s}. \tag{3.12}$$

We then substitute for the initial conditions in the above equation and solve for $\psi(s,u)$, this yields

$$\psi(s,u) = \frac{3u^2}{s(s^2+4u^2)} + \frac{s}{s^2+4u^2} + \frac{5u}{s^2+4u^2}. \tag{3.13}$$

Applying the inverse natural transform on both sides of (3.13). We note that the inverse natural transform of $\frac{3u^2}{s(s^2+4u^2)}$ can be got from the convolution theorem provided in table 3.1 or through partial fractions[2]. The inverse natural transforms of $\frac{s}{s^2+4u^2}$ and $\frac{5u}{s^2+4u^2}$ can be easily identified in 3.1. Thus (3.13) becomes

$$y(t) = \frac{3}{4} + \frac{1}{4}\cos(2t) + \frac{5}{2}\sin(2t). \tag{3.14}$$

This is the same solution that was obtained in [17] using the Laplace transform method.

[2]See [17] for a discussion on partial fractions

3.3 Natural transforms of the fractional derivatives

In this section, we are going to use theorem 3.2.1 (3.6) to deduce the natural transforms of the fractional derivatives (Caputo and Caputo-Fabrizio) and the Mittag-Leffler function. We will give an example to illustrate the use of the natural transform method to solve an ordinary fractional differential equation.

We first give Laplace transforms of the fractional derivatives and the Mittag-Leffler function before considering the natural transforms.

Definition 3.3.1 *The Laplace transform of the Caputo fractional derivative of order μ of the function $y(t)$ is defined as [22, 25, 36],*

$$\mathcal{L}[\mathcal{D}_t^{\mu} y(t)] = s^{\mu} Y(s) - \sum_{m=0}^{p-1} \frac{1}{s} s^{\mu-m} y^{(m)}(0), \quad p \in \mathbb{N}, \qquad \mu \in (p-1; p]. \tag{3.15}$$

Definition 3.3.2 *The Laplace Transform of the Mittag-Leffler function in two parameters is given as [26],*

$$\mathcal{L}[t^{\nu-1} E_{\mu,\nu}(\mp c^2 t^{\mu})] = Y(s) = \frac{s^{\mu-\nu}}{s^{\mu} \pm c^2}, \quad \mu, \nu > 0, \quad c \in \mathbb{R}, \quad |c| < s^{\mu}. \tag{3.16}$$

Definition 3.3.3 *The Laplace transform of the Caputo-Fabrizio fractional derivative of $y(t)$ of order $\alpha + p$, $p \in \mathbb{N} \cup 0$, is given as [12],*

$$\mathcal{L}[\mathbb{D}_t^{\alpha+p} y(t)] = \frac{1}{s - \alpha(s-1)} \left[s^{p+1} Y(s) - \sum_{m=0}^{p} s^{p-m} y^{(m)}(0) \right], \quad \alpha \in (0; 1]. \tag{3.17}$$

We state and prove the natural transforms of the fractional derivatives (Caputo and Caputo-Fabrizio) and the Mittag-Leffler function in the theorems below.

Theorem 3.3.1 *The natural transform of the Caputo fractional derivative of order μ of the function $y(t)$ is given by*

$$\mathcal{N}[\mathcal{D}_t^{\mu} y(t)] = \left(\frac{s}{u}\right)^{\mu} \psi(s,u) - \sum_{m=0}^{p-1} \frac{1}{s} \left(\frac{s}{u}\right)^{\mu-m} y^{(m)}(0), \quad p \in \mathbb{N}, \qquad \mu \in (p-1; p]. \tag{3.18}$$

Proof

Using (3.6) on (3.15), we have

$$\begin{aligned}
\mathcal{N}[\mathcal{D}_t^{\mu} y(t)] &= \frac{1}{u}\left[\left(\frac{s}{u}\right)^{\mu} Y\left(\frac{s}{u}\right) - \sum_{m=0}^{p-1} \frac{u}{s}.\left(\frac{s}{u}\right)^{\mu-m} y^{(m)}(0)\right] \\
&= \left(\frac{s}{u}\right)^{\mu} .\frac{1}{u} Y\left(\frac{s}{u}\right) - \sum_{m=0}^{p-1} \frac{1}{s}.\left(\frac{s}{u}\right)^{\mu-m} y^{(m)}(0) \\
&= \left(\frac{s}{u}\right)^{\mu} \mathcal{N}[y(t)] - \sum_{m=0}^{p-1} \frac{1}{s}\left(\frac{s}{u}\right)^{\mu-m} y^{(m)}(0) \\
&= \left(\frac{s}{u}\right)^{\mu} \psi(s,u) - \sum_{m=0}^{p-1} \frac{1}{s}\left(\frac{s}{u}\right)^{\mu-m} y^{(m)}(0).
\end{aligned}$$

Theorem 3.3.2 *The natural transform of the two parameter Mittag-Leffler function is*

$$\mathcal{N}[t^{\nu-1} E_{\mu,\nu}(\mp c^2 t^{\mu})] = \psi(s,u) = \frac{s^{\mu-\nu}}{u^{1-\nu}(s^{\mu} \pm c^2 u^{\mu})}. \tag{3.19}$$

Proof Using (3.6) on (3.16), we get

$$\mathcal{N}[t^{\nu-1} E_{\mu,\nu}(\mp c t^{\mu})] = \frac{1}{u}\left[\left(\frac{s}{u}\right)^{\mu-\nu} .\frac{1}{(\frac{s}{u})^{\mu} + c}\right] = \frac{s^{\mu-\nu}}{u^{1-\nu}(s^{\mu} \pm c u^{\mu})}.$$

Theorem 3.3.3 *The natural transform of the Caputo-Fabrizio fractional derivative of $y(t)$ of order $\alpha + p$, $p \in \mathbb{N} \cup 0$, is given as*

$$\mathcal{N}[\mathbb{D}^{\alpha+p} y(t)] = \frac{1}{s - \alpha(s-u)}\left[s\left(\frac{s}{u}\right)^{p} \psi(s,u) - \sum_{m=0}^{p} \left(\frac{s}{u}\right)^{p-m} y^{(m)}(0)\right], \quad \alpha \in (0;1]. \tag{3.20}$$

Proof

Using (3.6) on (3.17), we have

$$\begin{aligned}
\mathcal{N}[\mathbb{D}^{\alpha+p} y(t)] &= \frac{1}{u}\left[\left(\frac{s}{u} - \alpha(\frac{s}{u} - 1)\right)^{-1}\left(\frac{s^{p+1}}{u^{p+1}} Y\left(\frac{s}{u}\right) - \sum_{m=0}^{p} \frac{s^{p-m}}{u^{p-m}} y^{(m)}(0)\right)\right] \\
&= \frac{1}{s-\alpha(s-u)}\left[\frac{s^{p+1}}{u^{p}}.\frac{1}{u} Y\left(\frac{s}{u}\right) - \sum_{m=0}^{p} \left(\frac{s}{u}\right)^{p-m} y^{(m)}(0)\right] \\
&= \frac{1}{s-\alpha(s-u)}\left[s\left(\frac{s}{u}\right)^{p} \psi(s,u) - \sum_{m=0}^{p} \left(\frac{s}{u}\right)^{p-m} y^{(m)}(0)\right].
\end{aligned}$$

Corollary 3.3.1 *If we substitute $u = 1$ in the natural transforms of the fractional derivatives (Caputo and Caputo-Fabrizio) and the Mittag-Leffler function, we get the corresponding Laplace transforms.*

We note that the Caputo and Caputo-Fabrizio will have the same order of the fractional derivative if $\mu = \alpha + p$.

We now solve the fractional linear ordinary differential equation using the natural transform method.

Example 3.3 *Consider the fractional version of* (3.11),

$$\begin{aligned} &\mathcal{D}_t^{\mu} y(t) + 4y(t) = 3, \quad \mu \in (1, 2] \qquad (3.21)\\ &y(0) = 1, \quad y'(0) = 5. \end{aligned}$$

We consider both the Caputo and Caputo-Fabrizio fractional derivatives when solving (3.21) using the natural transform method, we start with the Caputo.

Taking the natural transform on both sides of (3.21),

$$\mathcal{N}[\mathcal{D}_t^{\mu} y(t)] + \mathcal{N}[4y(t)] = \mathcal{N}[3],$$

$$\left(\frac{s}{u}\right)^{\mu} \psi(s,u) - \frac{1}{s}\left(\frac{s}{u}\right)^{\mu} y(0) - \frac{1}{s}\left(\frac{s}{u}\right)^{\mu-1} y'(0) + 4\psi(s,u) = \frac{3}{s}.$$

Utilising the initial conditions and then solving for $\psi(s,u)$ in the above equation gives,

$$\psi(s,u) = \frac{3u^{\mu}}{s(s^{\mu} + 4u^{\mu})} + \frac{s^{\mu-1}}{s^{\mu} + 4u^{\mu}} + \frac{5us^{\mu-2}}{s^{\mu} + 4u^{\mu}}. \qquad (3.22)$$

Applying the convolution theorem or partial fractions to the first term on the right hand side above and simplifying yields,

$$\psi(s,u) = \frac{3}{4s} + \frac{s^{\mu-1}}{4(s^{\mu} + 4u^{\mu})} + \frac{5us^{\mu-2}}{s^{\mu} + 4u^{\mu}}. \qquad (3.23)$$

We then take the inverse natural transform on both sides of (3.23),

$$\mathcal{N}^{-1}\left[\psi(s,u)\right] = \mathcal{N}^{-1}\left[\frac{3}{4s}\right] + \mathcal{N}^{-1}\left[\frac{s^{\mu-1}}{4(s^{\mu} + 4u^{\mu})}\right] + \mathcal{N}^{-1}\left[\frac{5us^{\mu-2}}{s^{\mu} + 4u^{\mu}}\right]. \qquad (3.24)$$

In view of (3.4), table 3.1 and (3.19), from (3.24) we have,

$$y(t) = \frac{3}{4} + \frac{1}{4}E_{\mu,1}(-4t^{\mu}) + 5tE_{\mu,2}(-4t^{\mu}). \tag{3.25}$$

(3.25) is the general solution of (3.21). In view of (2.5) and (2.6) if $\mu = 2$, then (3.25) becomes

$$\begin{aligned} y(t) &= \frac{3}{4} + \frac{1}{4}E_{2,1}(-4t^2) + 5tE_{2,2}(-4t^2) \\ &= \frac{3}{4} + \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-4t^2)^k}{\Gamma(2k+1)} + 5t\sum_{k=0}^{\infty}\frac{(-4t^2)^k}{\Gamma(2k+2)} \\ &= \frac{3}{4} + \frac{1}{4}\cos(2t) + \frac{5}{2}\sin(2t). \end{aligned} \tag{3.26}$$

We note that (3.26) is the same as (3.14), thus indeed if $\mu = 2$ the fractional differential equation (3.21) coincides the integer order differential equation (3.11).

We now solve (3.21), but this time we consider the Caputo-Fabrizio fractional derivative, thus we have,

$$\mathbb{D}_t^{\mu}y(t) + 4y(t) = 3, \quad \mu \in (1,2] \tag{3.27}$$

$$y(0) = 1, \quad y'(0) = 5.$$

We note that $\mu = \alpha + 1$ with $\alpha \in (0,1]$ in this case. Applying the natural transform on both sides of (3.27),

$$\mathcal{N}[\mathbb{D}_t^{\mu}y(t)] + \mathcal{N}[4y(t)] = \mathcal{N}[3]$$

$$\mathcal{N}[\mathbb{D}_t^{\alpha+1}y(t)] + \mathcal{N}[4y(t)] = \mathcal{N}[3]$$

$$\frac{1}{s-\alpha(s-u)}\left[\frac{s^2}{u}\psi(s,u) - \frac{s}{u}y(0) - y'(0)\right] + 4\psi(s,u) = \frac{3}{s}.$$

Solving for $\psi(s,u)$ from above gives,

$$\psi(s,u) = \frac{3(s-\alpha(s-u))}{s^3 + 4us(s-\alpha(s-u))} + \frac{s+5}{s^2 + 4u(s-\alpha(s-u))}. \tag{3.28}$$

With the aid of MATHEMATICA, we take the inverse natural transform of (3.28) on both sides,

$$\mathcal{N}^{-1}\left[\psi(s,u)\right] = \mathcal{N}^{-1}\left[\frac{3(s-\alpha(s-u))}{s^3 + 4us(s-\alpha(s-u))}\right] + \mathcal{N}^{-1}\left[\frac{s+5}{s^2 + 4u(s-\alpha(s-u))}\right]. \tag{3.29}$$

(3.29) yields,

$$\begin{aligned}
y(t) &= \frac{3}{4}+\frac{1}{8}\exp(-2t(1-\alpha+\sqrt{1-3\alpha+\alpha^2})) \\
&+ \frac{1}{8}\exp(4t(\sqrt{1-3\alpha+\alpha^2})-2t(1-\alpha+\sqrt{1-3\alpha+\alpha^2})) \\
&- \frac{9\exp(-2t(1-\alpha+\sqrt{1-3\alpha+\alpha^2}))}{8\sqrt{1-3\alpha+\alpha^2}} \\
&+ \frac{9\exp(4t(\sqrt{1-3\alpha+\alpha^2})-2t(1-\alpha+\sqrt{1-3\alpha+\alpha^2}))}{8\sqrt{1-3\alpha+\alpha^2}} \\
&- \frac{\alpha\exp(-2t(1-\alpha+\sqrt{1-3\alpha+\alpha^2}))}{8\sqrt{1-3\alpha+\alpha^2}} \\
&+ \frac{\alpha\exp(4t(\sqrt{1-3\alpha+\alpha^2})-2t(1-\alpha+\sqrt{1-3\alpha+\alpha^2}))}{8\sqrt{1-3\alpha+\alpha^2}}. \qquad (3.30)
\end{aligned}$$

(3.30) is the general solution of (3.27), if $\alpha = 1$ in (3.30) then,

$$\begin{aligned}
y(t) &= \frac{3}{4}+\left(\frac{1}{8}+\frac{5i}{4}\right)\exp(-2it)+\left(\frac{1}{8}-\frac{5i}{4}\right)\exp(2it) \\
&= \frac{3}{4}+\frac{1}{4}\left(\frac{\exp(-2it)+\exp(2it)}{2}\right)+\frac{5}{2}i\left(\frac{\exp(-2it)-\exp(2it)}{2}\right) \qquad (3.31)
\end{aligned}$$

Using the trigonometric identities in appendix B, the above equation simplifies to,

$$y(t) = \frac{3}{4}+\frac{1}{4}\cos(2t)+\frac{5}{2}\sin(2t). \qquad (3.32)$$

We note that (3.32) is equal to (3.26), that's the Caputo and Caputo-Fabrizio fractional derivatives give the same solutions for $\mu = 2$, that is $\alpha = 1$. This is the same solution that we got when we solved the integer order ordinary differential equation (3.11).

In figures 3.1-3.4 we show how the solutions (3.25) and (3.30) change with time for different values of μ and α.

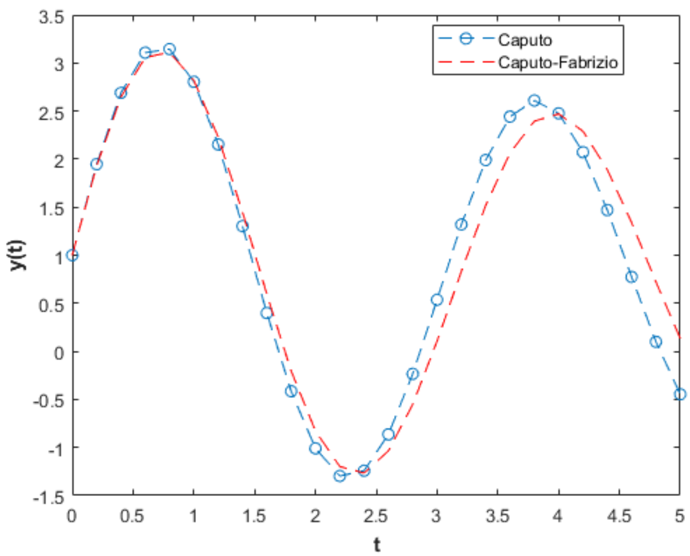

Figure 3.1: A plot of (3.25) and (3.30) for $\mu = 1.95$ and $\alpha = 0.95$.

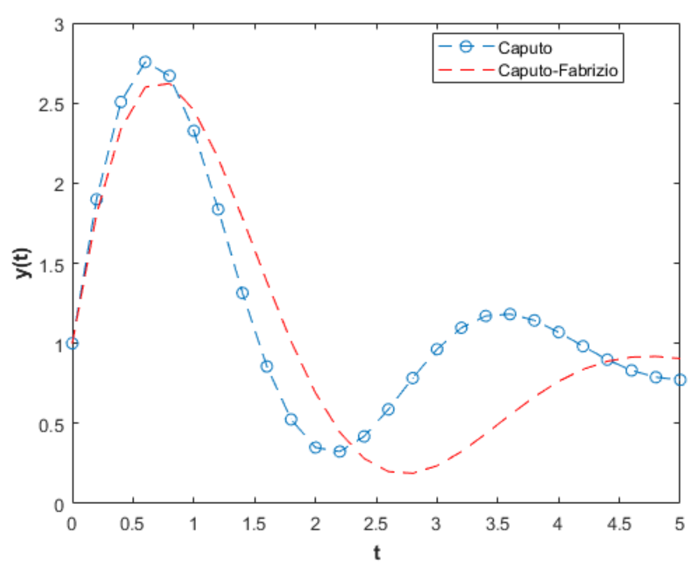

Figure 3.2: A plot of (3.25) and (3.30) for $\mu = 1.70$ and $\alpha = 0.70$.

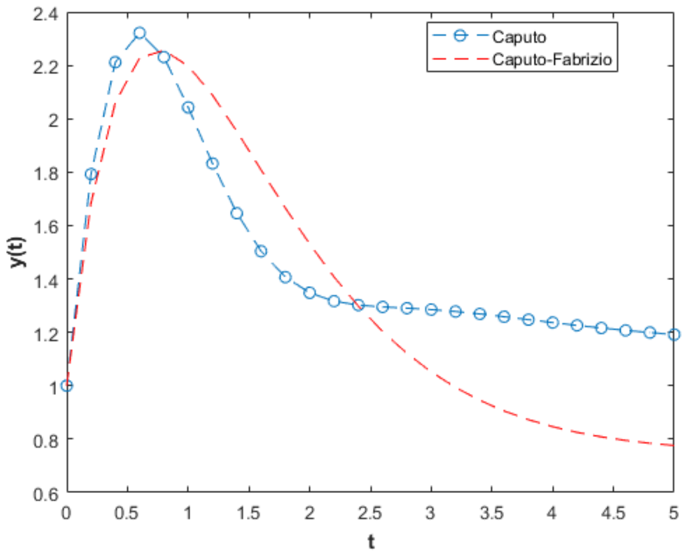

Figure 3.3: A plot of (3.25) and (3.30) for $\mu = 1.40$ and $\alpha = 0.40$.

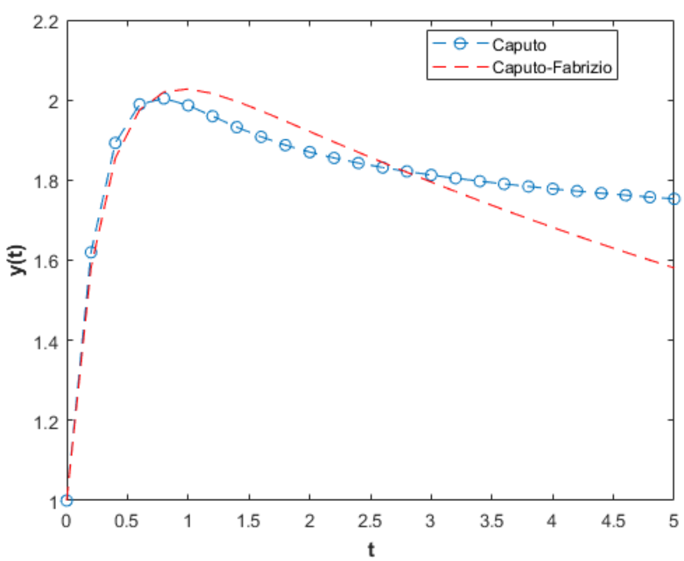

Figure 3.4: A plot of (3.25) and (3.30) for $\mu = 1.10$ and $\alpha = 0.10$.

In the figures 3.1-3.2 for the values of μ close to 2, the Caputo and Caputo-Fabrizio fractional derivatives have almost similar results.
However as the value of μ approaches 1, the Caputo fractional derivative tends to be affected more by the past than the Caputo-Fabrizio fractional derivative. This is particularly evident in figures 3.3 and 3.4 for large values of t.

Chapter 4

Decomposition Methods for Fractional Differential Equations

4.1 Introduction

Decomposition methods are one of the mathematical tools that have been used with success in the past two decades. They have the capability to handle with success a wide variety of both linear and non linear differential equations. The main advantage of the decomposition methods is that they are applied directly to the differential equation to be solved, there are no restrictive assumptions imposed on the problem in an attempt to simplify it. This means the original problem is not altered, that's the results obtained through the decomposition methods are considered to be of a realistic nature [3].

In this context, the term decomposition shall refer to expressing our solution of the differential equation as an infinite series [3]. Although the solution from the decomposition methods is in the form of an infinite series, in most cases the first few terms of the series are adequate to give an accurate solution [3].

In this chapter we focus on three decomposition methods, the Adomian decomposition method (ADM), the homotopy perturbation method (HPM) and an iterative method suggested by Daftardar and Jafari hereafter called the Daftardar-Jafari method (DJM).

Adomian decomposition method was developed by George Adomian, an extensive explanation of this method is given in [3], the use of this method was demonstrated in [3] by solving several nonlinear ordinary and partial differential equations. The ADM was used in [42] to solve the Fokker-Planck and the backward Kolmogorov equations, in [21] to tackle a system of nonlinear differential equations of fractional order.

The homotopy perturbation method was developed by Ji Huan He, this method is a result of the combination of the long established perturbation methods and a concept from topology known as homotopy [7]. In [7] the authors provided some helpful guidelines on constructing a homotopy equation, they went on to solve time dependent differential equations using the HPM. In [18] the HPM was applied to a system of nonlinear one and two dimensional Burgers equation and a system of Laplace's equation, in [38] it was used to solve Volterra and Fredholm integral equations.

An iterative method was suggested by Daftardar and Jafari hereafter called the Daftardar-Jafari method (DJM) in the year 2005 [14]. The effectiveness of this method was demonstrated when it was used to solve functional equations in [14]. Thereafter, the popularity of the DJM had to spread fast and it has been used extensively in different fields. The method was also used in [11] to obtain the solution of the Fisher's equation and it was applied to nonlinear fractional order diffusion and wave equations in [13].

In the next sections we will demonstrate how the ADM, HPM and the DJM can be used to solve a non linear ordinary fractional differential equation.

We will consider the following type of a non linear ordinary fractional differential equation,

$$\mathcal{D}_t^{\mu} y(t) + Ry(t) + Gy(t) = f(t), \quad \mu \in (p-1, p], \quad p \in \mathbb{N}, \tag{4.1}$$

with initial conditions,

$$y^{(m)}(0) = \frac{d^m y(0)}{dt^m}, \quad m = 0, 1, 2, ..., p-1. \tag{4.2}$$

$\mathcal{D}_t^{\mu}$ is the Caputo fractional derivative with respect to t, R is the linear operator, $Gy(t)$ represents nonlinear terms and $f(t)$ is a known function.

The solution of (4.1)-(4.2) according to the ADM, HPM and the DJM is an infinite series,

$$y(t) = \sum_{n=0}^{\infty} y_n(t). \tag{4.3}$$

It is important at this stage to mention that since the solution to (4.1)-(4.2) is expressed as an infinite series, for practical purposes we have to terminate the infinite series at a certain point [3]. An $n + 1$ term approximate solution to (4.1)-(4.2) will be given as [3]

$$\phi_n(t) = \sum_{i=0}^{n} y_i(t). \tag{4.4}$$

4.2 Adomian decomposition method

In this section we focus on the general description of the methodology of the ADM.

The first step is to take the fractional integral I_t^{μ} on both sides of (4.1)

$$I_t^{\mu}\mathcal{D}_t^{\mu}y(t) + I_t^{\mu}Ry(t) + I_t^{\mu}Gy(t) = I_t^{\mu}f(t), \tag{4.5}$$

in view of the properties (property iii) of the fractional integral discussed in Chapter 2, (4.5) can be written as

$$y(t) = \sum_{m=0}^{p-1} \frac{t^m}{m!}y^{(m)}(0) + I_t^{\mu}f(t) - I^{\mu}[Ry(t)] - I_t^{\mu}[Gy(t)]. \tag{4.6}$$

In the second step we decompose $y(t)$ into an infinite series as in (4.3) and the non linear term $Gy(t)$ is written as,

$$Gy(t) = \sum_{n=0}^{\infty} A_n(t), \tag{4.7}$$

where $A_n(t)$ are the Adomian polynomials given as [33],

$$A_n(t) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}G\left(\sum_{i=0}^{n} y_i(t)\lambda^i\right)\right]_{\lambda=0}. \tag{4.8}$$

We now substitute (4.3) and (4.7) into (4.6) to get,

$$\sum_{n=0}^{\infty} y_n(t) = \sum_{m=0}^{p-1} \frac{t^m}{m!}y^{(m)}(0) + I_t^{\mu}[f(t)] - I_t^{\mu}\left[R\sum_{n=0}^{\infty} y_n(t))\right] - I_t^{\mu}\left[\sum_{n=0}^{\infty} A_n(t)\right]. \tag{4.9}$$

The following iterative scheme is deduced from (4.9),

$$
\begin{aligned}
y_0(t) &= \sum_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0) + I_t^{\mu}[f(t)] = \mathcal{K}(t), \\
y_n(t) &= -I_t^{\mu}[Ry_{n-1}(t)] - I_t^{\mu}[A_{n-1}], \quad n = 1, 2, ...
\end{aligned}
$$

The approximate solution to (4.1)-(4.2) is then the summation of the terms obtained from above.

4.3 Homotopy Perturbation Method

This section focuses on the general description of the methodology of the HPM.

The first step in the HPM is to construct the homotopy of our equation that we intend to solve [7], in this case we construct the homotopy of (4.1) as,

$$
\mathcal{D}_t^{\mu} y(t) + q[Ry(t) + Gy(t)] = f(t), \quad q \in [0, 1], \tag{4.10}
$$

q is a perturbation parameter [7].
The integral representation of the above equation can be written as

$$
y(t) = \sum_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0) + I_t^{\mu} f(t) - q I_t^{\mu}[Ry(t) + Gy(t)]. \tag{4.11}
$$

The next step is to decompose our solution $y(t)$ as,

$$
y(t) = \sum_{n=0}^{\infty} q^n y_n(t), \tag{4.12}
$$

and the nonlinear term $Gy(t)$ is decomposed as,

$$
Gy(t) = \sum_{n=0}^{\infty} q^n A_n, \tag{4.13}
$$

where A_n is calculated as in (4.8). Some of the authors use the He's polynomials (H_n) in place of the Adomian polynomials (A_n), these two polynomials give the same results and their formulae is the same see [34].

We then substitute (4.12) and (4.13) into (4.11),

$$\sum_{n=0}^{\infty} q^n y_n(t) = \sum_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0) + I_t^\mu f(t) - q I_t^\mu \left[R \sum_{n=0}^{\infty} q^n y_n(t) + \sum_{n=0}^{\infty} q^n A_n \right].$$

We now compare the coefficients with the similar powers of q to get,

$$\begin{aligned} q^0 : y_0(t) &= \sum_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0) + I_t^\mu f(t) = \mathcal{K}(t), \\ q^n : y_n(t) &= -I_t^\mu \left[R y_{n-1}(t) + A_{n-1} \right], \quad n = 1, 2, ... \end{aligned}$$

The approximate solution to (4.1)-(4.2) is then the sum of the terms from the above iteration. We note the HPM gives the same solution as the ADM.

4.4 Daftardar-Jafari Method

In this section we give the general description of the methodology of the DJM.

In the first step we write (4.1)-(4.2) in integral form as,

$$y(t) = \sum_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0) + I_t^\mu f(t) - I_t^\mu [R y(t)] - I_t^\mu [G y(t)]. \tag{4.14}$$

In the second step we substitute (4.3) into the above equation, this gives

$$\sum_{n=0}^{\infty} y_n(t) = \sum_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0) + I_t^\mu f(t) - I_t^\mu \left[R \sum_{n=0}^{\infty} y_n(t) \right] - I_t^\mu \left[G \sum_{n=0}^{\infty} y_n(t) \right], \tag{4.15}$$

our non linear term is decomposed as [14],

$$G\left(\sum_{n=0}^{\infty} y_n(t) \right) = G(y_0) + \sum_{n=1}^{\infty} \left[G\left(\sum_{k=0}^{n} y_k(t) \right) - G\left(\sum_{k=0}^{n-1} y_k(t) \right) \right]. \tag{4.16}$$

Now, substituting the above equation into (4.15), we get

$$\sum_{n=0}^{\infty} y_n(t) = \mathcal{K}(t) - I_t^\mu \left[R \sum_{n=0}^{\infty} y_n(t) \right] - I_t^\mu \left[G(y_0) \right] - I_t^\mu \left[\sum_{n=1}^{\infty} \left[G\left(\sum_{k=0}^{n} y_k(t) \right) - G\left(\sum_{k=0}^{n-1} y_k(t) \right) \right] \right],$$

where $\mathcal{K}(t)$ is the term that is due to the initial condition and the known term $f(t)$.

Then from the above equation, the following recurrence relation is defined,

$$
\begin{aligned}
y_0 &= \mathcal{K}(t), \\
y_1 &= -I_t^{\mu}[Ry_0] - I_t^{\mu}[G(y_0)], \\
y_2 &= -I_t^{\mu}[Ry_1] - I_t^{\mu}[G(y_0 + y_1) - G(y_0)], \\
y_n &= -I_t^{\mu}[Ry_{n-1}] - I_t^{\mu}[G(y_0 + ... + y_{n-1}) - G(y_0 + ... + y_{n-2})], \quad n = 3, 4, ...
\end{aligned}
$$

The $n + 1$ term approximate solution to (4.1)-4.2 is then given by $y = y_0 + y_1 + ... + y_n$.

4.5 Convergence and rate of convergence of decomposition methods

In this section we give a brief discussion on the convergence and the rate of convergence of the series solutions of the ADM, HPM and DJM.

We state without proof the following theorem that guarantees convergence of the decomposition methods [19].

Theorem 4.5.1 *The infinite series* $\sum_{n=0}^{\infty} y_n(t)$ *converges to the analytical solution* $y(t)$ *of* (4.1)-(4.2) *whenever,* $\| y_n \| \leq \rho \| y_{n-1} \|, \quad \rho \in [0; 1), \quad n = 1, 2, ...$

The value of ρ is not only used to determine convergence, but can also be used to compare the rates of convergence of decomposition methods [19]. The lesser [3] the value of ρ, the higher the rate of convergence of the decomposition method [19].

The value of ρ is not necessarily the same for a series solution of a decomposition method, we give the following important definition [19],

Definition 4.5.1

$$
\rho_n = \begin{cases} \frac{\|y_n(t)\|}{\|y_{n-1}(t)\|}, & \| y_{n-1} \| \neq 0, \quad n = 1, 2, ...; \\ 0, & \| y_{n-1} \| = 0. \end{cases} \tag{4.17}
$$

[3] See [19] for a detailed discussion on rate of convergence.

Corollary 4.5.1 *In view of theorem 4.5.1, the series solution of the decomposition methods will converge to an analytical solution if* $\rho_n \in [0, 1)$, $n = 1, 2, ...$

The norms[4] in definition 4.5.1 are defined as [10],

$$\| y_n(t) \| = \sqrt{\int_a^b | y_n(t) |^2 \, dt}, \quad a, b \in \mathbb{R}, \quad b > a, \tag{4.18}$$

and

$$\| y_{n-1}(t) \| = \sqrt{\int_a^b | y_{n-1}(t) |^2 \, dt}, \quad a, b \in \mathbb{R}, \quad b > a. \tag{4.19}$$

4.6 Example and Comparisons of the methods

In this section we will solve a non linear fractional ordinary differential equation using each of the three decomposition methods that we have already studied and thereafter do a comparison of the results.

Example 4.1 *We consider the following non linear fractional ordinary differential equation [3],*

$$\mathcal{D}_t^{\mu} x(t) + x^2(t) = t^2 + 1, \quad 0 < \mu \leq 1 \tag{4.20}$$
$$x(0) = 0,$$

$\mathcal{D}_t^{\mu}$ *is the Caputo fractional derivative.*

We now solve (4.20) using the three decomposition methods, ADM, HPM and the DJM. In view of (4.6), the integral representation of (4.20) is

$$x(t) = \frac{t^{\mu}}{\Gamma(\mu + 1)} + \frac{1}{\Gamma(\mu)} \left(\frac{2t^{\mu+2}}{2\mu + 3\mu^2 + \mu^3} \right) - I_t^{\mu}[x^2(t)]. \tag{4.21}$$

We start with the ADM, in view of (4.3) and (4.7) we have,

$$x_n(t) = \sum_{n=0}^{\infty} x_n(t), \tag{4.22}$$

[4] See [10] for a detailed discussion on the norms and their calculations

and

$$x^2(t) = \sum_{n=0}^{\infty} A_n(t). \tag{4.23}$$

Substituting (4.22) and (4.23) into (4.21)gives,

$$\sum_{n=0}^{\infty} x_n(t) = \frac{t^\mu}{\Gamma(\mu+1)} + \frac{1}{\Gamma(\mu)}\left(\frac{2t^{\mu+2}}{2\mu+3\mu^2+\mu^3}\right) - I_t^\mu\left[\sum_{n=0}^{\infty} A_n(t)\right]. \tag{4.24}$$

Thus from (4.24),

$$\begin{aligned} x_0(t) &= \frac{t^\mu}{\Gamma(\mu+1)} + \frac{1}{\Gamma(\mu)}\left(\frac{2t^{\mu+2}}{2\mu+3\mu^2+\mu^3}\right), && (4.25)\\ x_n(t) &= -I_t^\mu[A_{n-1}], \quad n = 1,2,3,... && (4.26)\end{aligned}$$

The subsequent terms after $x_0(t)$ are obtained from (4.26) as follows,

$$\begin{aligned} x_1(t) &= -I_t^\mu[A_0] \\ &= -\frac{t^{3\mu}\Gamma(2\mu+1)}{\Gamma(3\mu+1)\Gamma(\mu+1)^2} - \frac{4t^{3\mu+2}\Gamma(2\mu+3)}{\Gamma(\mu+3)\Gamma(\mu+1)\Gamma(3\mu+3)} - \frac{4t^{3\mu+4}\Gamma(2\mu+5)}{\Gamma(3\mu+5)\Gamma(\mu+3)^2}, \\ x_2(t) &= -I_t^\mu[A_1] \\ &= \frac{2\Gamma(2\mu+1)\Gamma(4\mu+1)t^{5\mu}}{\Gamma(\mu+1)^3\Gamma(3\mu+1)\Gamma(5\mu+1)} + \frac{8\Gamma(2\mu+3)\Gamma(4\mu+3)t^{5\mu+2}}{\Gamma(5\mu+3)\Gamma(\mu+1)^2\Gamma(\mu+3)\Gamma(3\mu+3)} \\ &+ \frac{8\Gamma(2\mu+5)\Gamma(4\mu+5)t^{5\mu+4}}{\Gamma(5\mu+5)\Gamma(\mu+1)\Gamma(\mu+3)^2\Gamma(3\mu+5)} + \frac{4\Gamma(2\mu+1)\Gamma(4\mu+3)t^{5\mu+2}}{\Gamma(5\mu+3)\Gamma(\mu+1)^2\Gamma(\mu+3)\Gamma(3\mu+1)} \\ &+ \frac{16\Gamma(2\mu+3)\Gamma(4\mu+5)t^{5\mu+4}}{\Gamma(5\mu+5)\Gamma(\mu+1)\Gamma(\mu+3)^2\Gamma(3\mu+3)} + \frac{16\Gamma(2\mu+5)\Gamma(4\mu+7)t^{5\mu+6}}{\Gamma(5\mu+7)\Gamma(\mu+3)^3\Gamma(3\mu+5)}. \end{aligned}$$

We note that the MATHEMATICA code for calculating the Adomian polynomials $A_0, A_1, A_2, ...$ is provided in appendix A.

The three term approximate solution to (4.20) is then given by $x(t) = x_0(t) + x_1(t) + x_2(t)$.

Solving (4.20) using the HPM, we start in the same way as in the ADM. The integral representation of (4.20) is (4.21). We then construct the homotopy of (4.21) as,

$$x(t) = \frac{t^\mu}{\Gamma(\mu+1)} + \frac{1}{\Gamma(\mu)}\left(\frac{2t^{\mu+2}}{2\mu+3\mu^2+\mu^3}\right) - qI_t^\mu[x^2(t)], \quad q \in [0;1]. \tag{4.27}$$

In view of (4.12) and (4.13), we have

$$x(t) = \sum_{n=0}^{\infty} q^n x_n(t) \tag{4.28}$$

and

$$x^2(t) = \sum_{n=0}^{\infty} q^n A_n(t). \tag{4.29}$$

Substituting (4.28) and (4.29) into (4.27) yields,

$$\sum_{n=0}^{\infty} q^n x_n(t) = \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{1}{\Gamma(\mu)}\left(\frac{2t^{\mu+2}}{2\mu+3\mu^2+\mu^3}\right) - qI_t^{\mu}\left[\left[\sum_{n=0}^{\infty} q^n A_n(t)\right]\right]. \tag{4.30}$$

Then from (4.30) we compare terms with the same powers of q,

$$\begin{aligned}
q^0 : x_0(t) &= \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{1}{\Gamma(\mu)}\left(\frac{2t^{\mu+2}}{2\mu+3\mu^2+\mu^3}\right),\\
q^1 : x_1(t) &= -I^{\mu}[A_0]\\
&= -\frac{t^{3\mu}\Gamma(2\mu+1)}{\Gamma(3\mu+1)\Gamma(\mu+1)^2} - \frac{4t^{3\mu+2}\Gamma(2\mu+3)}{\Gamma(\mu+3)\Gamma(\mu+1)\Gamma(3\mu+3)} - \frac{4t^{3\mu+4}\Gamma(2\mu+5)}{\Gamma(3\mu+5)\Gamma(\mu+3)^2}\\
q^2 : x_2(t) &= -I^{\mu}[A_1],\\
&= \frac{2\Gamma(2\mu+1)\Gamma(4\mu+1)t^{5\mu}}{\Gamma(\mu+1)^3\Gamma(3\mu+1)\Gamma(5\mu+1)} + \frac{8\Gamma(2\mu+3)\Gamma(4\mu+3)t^{5\mu+2}}{\Gamma(5\mu+3)\Gamma(\mu+1)^2\Gamma(\mu+3)\Gamma(3\mu+3)}\\
&+ \frac{8\Gamma(2\mu+5)\Gamma(4\mu+5)t^{5\mu+4}}{\Gamma(5\mu+5)\Gamma(\mu+1)\Gamma(\mu+3)^2\Gamma(3\mu+5)} + \frac{4\Gamma(2\mu+1)\Gamma(4\mu+3)t^{5\mu+2}}{\Gamma(5\mu+3)\Gamma(\mu+1)^2\Gamma(\mu+3)\Gamma(3\mu+1)}\\
&+ \frac{16\Gamma(2\mu+3)\Gamma(4\mu+5)t^{5\mu+4}}{\Gamma(5\mu+5)\Gamma(\mu+1)\Gamma(\mu+3)^2\Gamma(3\mu+3)} + \frac{16\Gamma(2\mu+5)\Gamma(4\mu+7)t^{5\mu+6}}{\Gamma(5\mu+7)\Gamma(\mu+3)^3\Gamma(3\mu+5)}.
\end{aligned}$$

Our approximate solution to (4.20) according to the HPM is then given by the sum of the terms from the above iteration, we note that this is the same solution obtained using the ADM.

We then use the DJM to solve (4.20), we start from (4.21) which is the integral representation of (4.20). In this case our non linear term is $G(x(t)) = x^2(t)$, thus in view of (4.3) and (4.16) we have,

$$\begin{aligned}
\sum_{n=0}^{\infty} x_n(t) &= \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{1}{\Gamma(\mu)}\left(\frac{2t^{\mu+2}}{2\mu+3\mu^2+\mu^3}\right)\\
&- I_t^{\mu}[G(x_0)] - I_t^{\mu}\left[\sum_{n=1}^{\infty}\left[G\left(\sum_{k=0}^{n} x_k\right) - G\left(\sum_{k=0}^{n-1} x_k\right)\right]\right].
\end{aligned} \tag{4.31}$$

Then the following terms are deduced from (4.31),

$$\begin{aligned}
x_0(t) &= \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{1}{\Gamma(\mu)}\left(\frac{2t^{\mu+2}}{2\mu+3\mu^2+\mu^3}\right),\\
x_1(t) &= -I_t^{\mu}[G(x_0(t))]\\
&= -\frac{t^{3\mu}\Gamma(2\mu+1)}{\Gamma(3\mu+1)\Gamma(\mu+1)^2} - \frac{4t^{3\mu+2}\Gamma(2\mu+3)}{\Gamma(\mu+3)\Gamma(\mu+1)\Gamma(3\mu+3)} - \frac{4t^{3\mu+4}\Gamma(2\mu+5)}{\Gamma(3\mu+5)\Gamma(\mu+3)^2},\\
x_2(t) &= -I_t^{\mu}[G(x_0+x_1) - G(x_0)]\\
&= \frac{2\Gamma(2\mu+1)\Gamma(4\mu+1)t^{5\mu}}{\Gamma(\mu+1)^3\Gamma(3\mu+1)\Gamma(5\mu+1)} + \frac{8\Gamma(2\mu+3)\Gamma(4\mu+3)t^{5\mu+2}}{\Gamma(5\mu+3)\Gamma(\mu+1)^2\Gamma(\mu+3)\Gamma(3\mu+3)}\\
&+ \frac{8\Gamma(2\mu+5)\Gamma(4\mu+5)t^{5\mu+4}}{\Gamma(5\mu+5)\Gamma(\mu+1)\Gamma(\mu+3)^2\Gamma(3\mu+5)} + \frac{4\Gamma(2\mu+1)\Gamma(4\mu+3)t^{5\mu+2}}{\Gamma(5\mu+3)\Gamma(\mu+1)^2\Gamma(\mu+3)\Gamma(3\mu+1)}\\
&+ \frac{16\Gamma(2\mu+3)\Gamma(4\mu+5)t^{5\mu+4}}{\Gamma(5\mu+5)\Gamma(\mu+1)\Gamma(\mu+3)^2\Gamma(3\mu+3)} + \frac{16\Gamma(2\mu+5)\Gamma(4\mu+7)t^{5\mu+6}}{\Gamma(5\mu+7)\Gamma(\mu+3)^3\Gamma(3\mu+5)}\\
&- \frac{\Gamma(2\mu+1)^2\Gamma(6\mu+1)t^{7\mu}}{\Gamma(7\mu+1)\Gamma(\mu+1)^4\Gamma(3\mu+1)^2} - \frac{8\Gamma(2\mu+1)\Gamma(2\mu+3)\Gamma(6\mu+3)t^{7\mu+2}}{\Gamma(7\mu+3)\Gamma(\mu+1)^3\Gamma(\mu+3)\Gamma(3\mu+1)\Gamma(3\mu+3)}\\
&- \frac{16\Gamma(2\mu+3)^2\Gamma(6\mu+5)t^{7\mu+4}}{\Gamma(7\mu+5)\Gamma(\mu+1)^2\Gamma(\mu+3)^2\Gamma(3\mu+3)^2} - \frac{32\Gamma(2\mu+3)\Gamma(2\mu+5)\Gamma(6\mu+7)t^{7\mu+6}}{\Gamma(7\mu+7)\Gamma(\mu+1)\Gamma(\mu+3)^3\Gamma(3\mu+3)\Gamma(3\mu+5)}\\
&- \frac{8\Gamma(2\mu+1)\Gamma(2\mu+5)\Gamma(6\mu+5)t^{7\mu+4}}{\Gamma(7\mu+5)\Gamma(\mu+1)^2\Gamma(\mu+3)^2\Gamma(3\mu+1)\Gamma(3\mu+5)} - \frac{16\Gamma(2\mu+5)^2\Gamma(6\mu+9)t^{7\mu+8}}{\Gamma(7\mu+9)\Gamma(\mu+3)^4\Gamma(3\mu+5)^2}.
\end{aligned}$$

The approximate solution to (4.20) is then given by $x(t) = x_0(t) + x_1 + x_2$. We note that the approximate solution using the DJM produces more terms than the approximate solution using the ADM/HPM. In fact, to be more specific the DJM approximate solution adds extra terms to the ADM/HPM approximate solution.

In Figures 4.1-4.5, we plot the approximate solutions of (4.20) that we obtained using the ADM, HPM and the DJM for different values of μ.

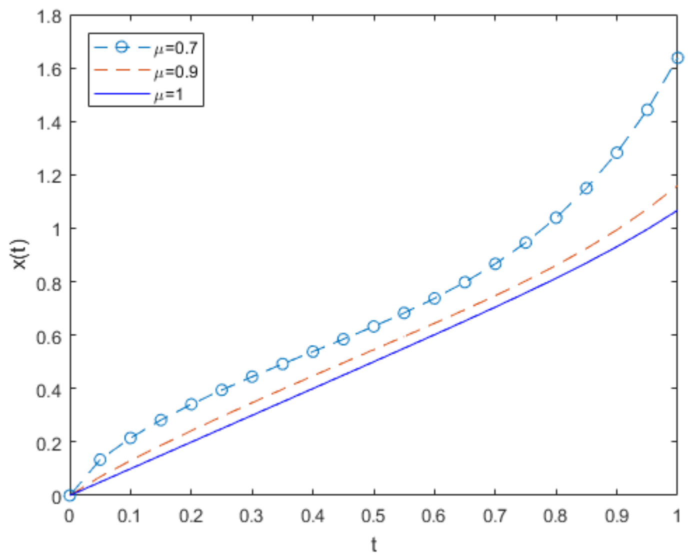

Figure 4.1: A plot of the ADM/HPM approximate solution to (4.20) for different values of μ.

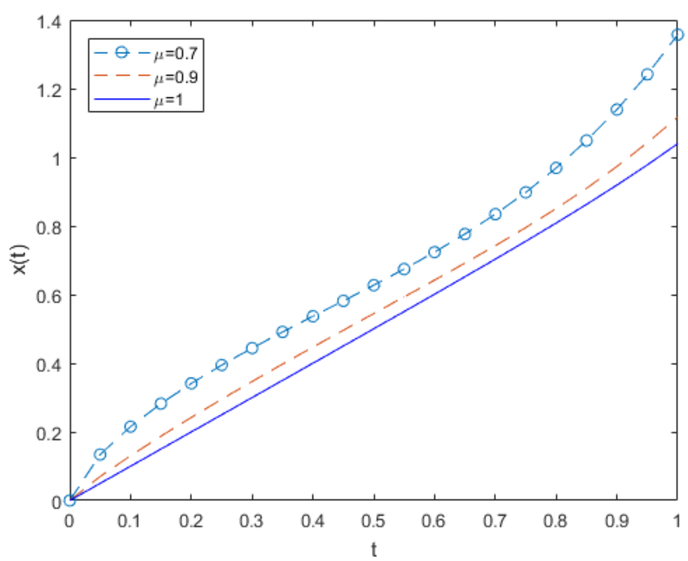

Figure 4.2: A plot of the DJM approximate solution to (4.20) for different values of μ.

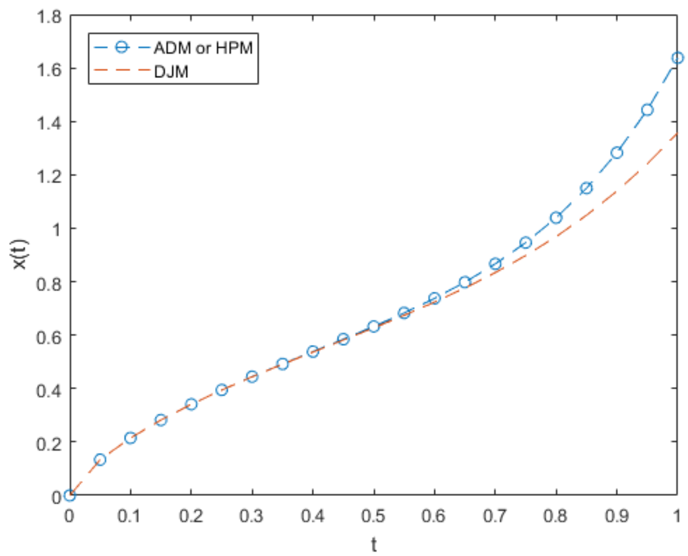

Figure 4.3: A plot of the ADM/HPM and DJM approximate solution to (4.20) for $\mu = 0.7$.

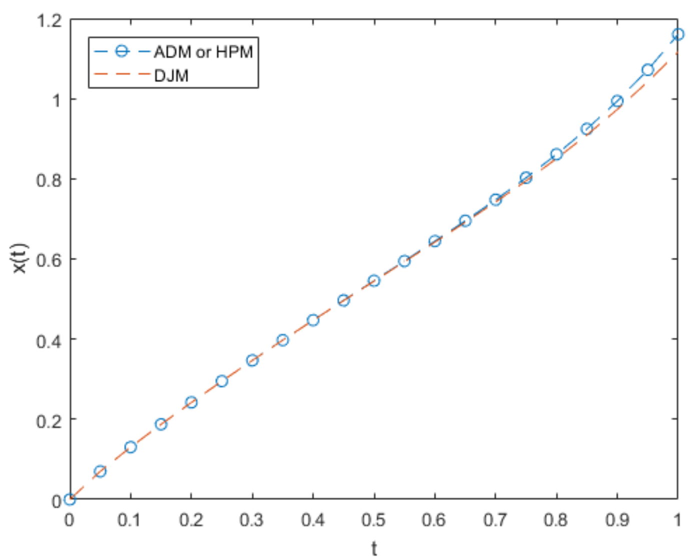

Figure 4.4: A plot of the ADM/HPM and DJM approximate solution to (4.20) for $\mu = 0.9$.

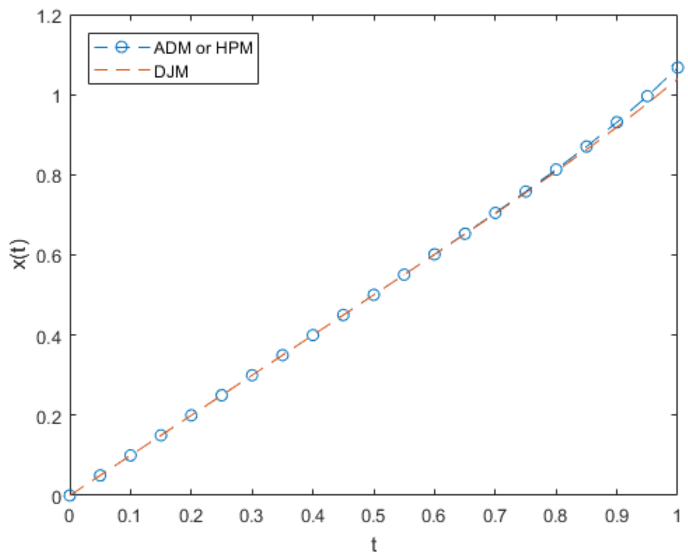

Figure 4.5: A plot of the ADM/HPM and DJM approximate solution to (4.20) for $\mu = 1$.

We note that from Figure 4.1-4.5, as the value $\mu \to 1$, the ADM/HPM and the DJM give solutions that are in close agreement, but as the value of μ is decreased from 1 the methods tend to give different solutions.

We also note that as the value of μ is decreased, the DJM shows a slower change in it's approximate solution, indicating some form of stability compared to the ADM/HPM. This is due to the extra terms that the DJM approximate solution produces in addition to the ADM/HPM approximate solution.

We now investigate the convergence and the rate of convergence of the three decomposition methods.

In view of (4.17), (4.18) and (4.19) we have,

$$\| x_n(t) \| = \sqrt{\int_0^1 | x_n(t) |^2 \, dt}, \tag{4.32}$$

$$\| x_{n-1}(t) \| = \sqrt{\int_0^1 | x_{n-1}(t) |^2 \, dt}, \tag{4.33}$$

and

$$\rho_n = \frac{\| x_n(t) \|}{\| x_{n-1}(t) \|}, \quad n = 1, 2, ... \tag{4.34}$$

We assign the values of ρ_n and $\hat{\rho}_n$ to the ADM/HPM and DJM respectively. We then compute the numerical values of ρ_n and $\hat{\rho}_n$ for different values of μ with the help of MATHEMATICA.

When $\mu = 1$,

$$\begin{aligned} \rho_1 &= \frac{\| x_1 \|}{\| x_0 \|} = 0.2437520234 < 1. \\ \rho_2 &= \frac{\| x_2 \|}{\| x_1 \|} = 0.3619159473 < 1. \end{aligned}$$

$$\begin{aligned} \hat{\rho}_1 &= \frac{\| x_1 \|}{\| x_0 \|} = 0.2437520234 < 1. \\ \hat{\rho}_2 &= \frac{\| x_2 \|}{\| x_1 \|} = 0.0321400786 < 1. \end{aligned}$$

When $\mu = 0.9$,

$$\begin{aligned} \rho_1 &= \frac{\| x_1 \|}{\| x_0 \|} = 0.313392 < 1. \\ \rho_2 &= \frac{\| x_2 \|}{\| x_1 \|} = 0.48882 < 1. \end{aligned}$$

$$\begin{aligned} \hat{\rho}_1 &= \frac{\| x_1 \|}{\| x_0 \|} = 0.313392 < 1. \\ \hat{\rho}_2 &= \frac{\| x_2 \|}{\| x_1 \|} = 0.416539 < 1. \end{aligned}$$

When $\mu = 0.7$,

$$\begin{aligned} \rho_1 &= \frac{\| x_1 \|}{\| x_0 \|} = 0.49737 < 1. \\ \rho_2 &= \frac{\| x_2 \|}{\| x_1 \|} = 0.851877 < 1. \end{aligned}$$

$$\begin{aligned} \hat{\rho}_1 &= \frac{\| x_1 \|}{\| x_0 \|} = 0.49737 < 1. \\ \hat{\rho}_2 &= \frac{\| x_2 \|}{\| x_1 \|} = 0.459523 < 1. \end{aligned}$$

When $\mu = 0.1$,

$$\begin{aligned}\rho_1 &= \frac{\| x_1 \|}{\| x_0 \|} = 1.35439 > 1.\\ \rho_2 &= \frac{\| x_2 \|}{\| x_1 \|} = 2.87693 > 1.\end{aligned}$$

$$\begin{aligned}\hat{\rho}_1 &= \frac{\| x_1 \|}{\| x_0 \|} = 1.35439 > 1.\\ \hat{\rho}_2 &= \frac{\| x_2 \|}{\| x_1 \|} = 0.764092 < 1.\end{aligned}$$

We note that using the ADM, HPM and the DJM to approximate the solution of (4.20) as $\mu \to 1$ all the three methods converge, but as $\mu \to 0$ the methods tend to diverge.

We also note that by comparing the values of ρ_2 and $\hat{\rho}_2$, the rate of convergence for the DJM is higher than that of the ADM/HPM, as we stated earlier that the lesser the value of ρ the higher the rate of convergence. This is clearly attributed to the extra terms in the approximate solution of DJM as compared to the ADM/HPM.

Chapter 5

Natural Transform Decomposition Method

5.1 Introduction

In this chapter, we combine the natural transform method and the decomposition methods to come up with one method namely, the natural transform decomposition method (NTDM). The natural transform method was dealt with in the third chapter and the decomposition methods were discussed in the previous chapter.

The originality of the NTDM and where it has been applied has already been dealt with in the literature review in the first chapter. Thus, in this chapter we are going to focus on the algorithm of the method and it's application to the fractional differential equations. In particular we will apply the NTDM to the non linear Klein-Gordon differential equation of fractional order, do convergence analysis of the method and compare our results with the Fractional reduced transform method (FRTM).

Since we are going to consider three decomposition methods that were discussed in the previous chapter, we are going to combine each of the decomposition method with the natural transform. This means we are going to have three forms of the NTDM namely, the Natural Adomian decomposition method (NADM) [1], the Natural homotopy perturbation method (NHPM) [34]. We will then propose our own form of the NTDM by combining the natural transform and the Daftardar-Jafari method (DJM), and we will refer to this method as the Natural Daftardar-Jafari method(NDJM).

In this chapter, our fractional derivative will be the Caputo, then in the next chapter we are going to deal with the Caputo-Fabrizio fractional derivative.

5.2 General description of the Natural transform decomposition method

Consider the following type of the time fractional partial differential equation,

$$\mathcal{D}_t^{\mu} y(\bar{x},t) + Ry(\bar{x},t) + Gy(\bar{x},t) = f(\bar{x},t), \quad p-1 < \mu \leq p, \quad p \in \mathbb{N}. \tag{5.1}$$

with initial conditions,

$$y^{(m)}(\bar{x},0) = \frac{\partial^m y(\bar{x},0)}{\partial t^m} = y_m(\bar{x}), \quad m = 0,1,2,...,p-1. \tag{5.2}$$

$\bar{x} = (x_1, x_2, ...)$, $\mathcal{D}_t^{\mu} = \frac{\partial^{\mu}}{\partial t^{\mu}}$ is the Caputo fractional derivative, R represents the linear operator, $Gy(\bar{x},t)$ is the non linear term and $f(\bar{x},t)$ is taken as the source term.

The first step is to take the natural transform with respect to t of (5.1) on both sides,

$$\mathcal{N}[D_t^{\mu} y(\bar{x},t)] + \mathcal{N}[Ry(\bar{x},t) + Gy(\bar{x},t)] = \mathcal{N}[f(\bar{x},t)], \tag{5.3}$$

we then substitute for $\mathcal{N}[D_t^{\mu} y(\bar{x},t)]$ using (3.18) and apply the given initial conditions, after doing this we solve for $\psi(\bar{x},s,u)$,

$$\psi(\bar{x},s,u) = \sum_{m=0}^{p-1} u^m s^{-(m+1)} y_m(\bar{x}) + \left(\frac{u}{s}\right)^{\mu} \mathcal{N}[f(\bar{x},t)] - \left(\frac{u}{s}\right)^{\mu} \mathcal{N}[Ry(\bar{x},t) + Gy(\bar{x},t)]. \tag{5.4}$$

The second step is to take the inverse natural transform of the above equation,

$$\begin{aligned} \mathcal{N}^{-1}[\psi(\bar{x},s,u)] &= \mathcal{N}^{-1}\left[\sum_{m=0}^{p-1} u^m s^{-(m+1)} y_m(\bar{x}) + \left(\tfrac{u}{s}\right)^{\mu} \mathcal{N}[f(\bar{x},t)]\right] \\ &\quad -\mathcal{N}^{-1}\left[\left(\tfrac{u}{s}\right)^{\mu} \mathcal{N}[Ry(\bar{x},t) + Gy(\bar{x},t)]\right]. \end{aligned} \tag{5.5}$$

This yields,

$$y(\bar{x},t) = \mathcal{K}(\bar{x},t) - \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[Ry(\bar{x},t) + Gy(\bar{x},t)]\right], \tag{5.6}$$

where $\mathcal{K}(\bar{x},t)$ is the term due to the initial conditions and the source term.

In the third step, we now apply the decomposition methods to (5.6), we will start with the ADM. We decompose $y(\bar{x},t)$ into

$$y(\bar{x},t) = \sum_{n=0}^{\infty} y_n(\bar{x},t), \tag{5.7}$$

and the non linear term is written as

$$Gy(\bar{x},t) = \sum_{n=0}^{\infty} A_n(\bar{x},t), \tag{5.8}$$

where $A_n(\bar{x},t)$ are the Adomian polynomials that were discussed in the previous chapter. Substituting (5.7) and (5.8) into (5.6) yields

$$\sum_{n=0}^{\infty} y_n(\bar{x},t) = \mathcal{K}(\bar{x},t) - \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}\left[R\sum_{n=0}^{\infty} y_n(\bar{x},t) + \sum_{n=0}^{\infty} A_n\right]\right]. \tag{5.9}$$

The following is then deduced from (5.9),

$$\begin{aligned} y_0(\bar{x},t) &= \mathcal{K}(\bar{x},t), \\ y_n(\bar{x},t) &= -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[Ry_{n-1}(\bar{x},t) + A_{n-1}(\bar{x},t)]\right], \quad n = 1,2,... \end{aligned} \tag{5.10}$$

The approximate solution to (5.1)-(5.2) is the summation of the terms from (5.10).

We now apply the HPM to (5.6), we construct the homotopy of (5.6) as [34],

$$y(\bar{x},t) = \mathcal{K}(\bar{x},t) - q\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[Ry(\bar{x},t) + Gy(\bar{x},t)]\right], \quad q \in [0,1]. \tag{5.11}$$

We then decompose $y(\bar{x},t)$ into

$$y(\bar{x},t) = \sum_{n=0}^{\infty} q^n y_n(\bar{x},t), \tag{5.12}$$

and the non linear term $Gy(\bar{x},t)$ is written as

$$Gy(\bar{x},t) = \sum_{n=0}^{\infty} q^n A_n(\bar{x},t), \tag{5.13}$$

where $A_n(\bar{x},t)$ are the Adomian polynomials. Substituting (5.12) and (5.13) into (5.11) gives,

$$\sum_{n=0}^{\infty} q^n y_n(\bar{x},t) = \mathcal{K}(\bar{x},t) - q\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}\left[R\sum_{n=0}^{\infty} q^n y_n(\bar{x},t) + \sum_{n=0}^{\infty} q^n A_n(\bar{x},t)\right]\right]. \tag{5.14}$$

We then compare terms with the same powers of q:

$$\begin{aligned} q^0 : y_0(\bar{x},t) &= \mathcal{K}(\bar{x},t), \\ q^n : y_n(\bar{x},t) &= -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[R(y_{n-1}(\bar{x},t)) + A_{n-1}]\right], \quad n = 1,2,... \end{aligned} \tag{5.15}$$

The approximate solution to (5.1)-(5.2) is then the sum of the terms from (5.15).

We now propose our own form of the NTDM, we combine the natural transform and the DJM, we were able to apply our proposed method in [24]. Starting from (5.6) we use the DJM, we substitute (5.7) into (5.6) to get

$$\sum_{n=0}^{\infty} y_n(\bar{x},t) = \mathcal{K}(\bar{x},t) - \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}\left[R\sum_{n=0}^{\infty} y_n(\bar{x},t) + G\sum_{n=0}^{\infty} y_n(\bar{x},t)\right]\right], \tag{5.16}$$

$G\left(\sum_{n=0}^{\infty} y_n(\bar{x},t)\right)$ is then written as [14],

$$G\left(\sum_{n=0}^{\infty} y_n(\bar{x},t)\right) = G(y_0(\bar{x},t)) + \sum_{n=1}^{\infty}\left[G\left(\sum_{k=0}^{n} y_k(\bar{x},t)\right) - G\left(\sum_{k=0}^{n-1} y_k(\bar{x},t)\right)\right]. \tag{5.17}$$

Substituting (5.17) into (5.16), we get

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(\bar{x},t) &= \mathcal{K}(\bar{x},t) - \mathcal{N}^{-1}\Bigg[\left(\tfrac{u}{s}\right)^{\mu}\mathcal{N}\Bigg[R\sum_{n=0}^{\infty} y_n(\bar{x},t) + G(y_0(\bar{x},t)) \\ &\quad + \sum_{n=1}^{\infty}\left[G\left(\sum_{k=0}^{n} y_k(\bar{x},t)\right) - G\left(\sum_{k=0}^{n-1} y_k(\bar{x},t)\right)\right]\Bigg]\Bigg]. \end{aligned} \tag{5.18}$$

The following iteration is then deduced from (5.18), for simplicity we shall denote $y_k(\bar{x},t)$ values as y_k,

$$\begin{aligned} y_0(\bar{x},t) &= \mathcal{K}(\bar{x},t), \\ y_1(\bar{x},t) &= -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[Ry_0(\bar{x},t) + G(y_0)]\right], \\ y_2(\bar{x},t) &= -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[Ry_1(\bar{x},t) + G(y_1+y_0) - G(y_0)]\right], \\ &\vdots \\ y_n(\bar{x},t) &= -\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[Ry_{n-1}(\bar{x},t) + G(y_0+...+y_{n-1}) - G(y_0+...+y_{n-2})]\right], \quad n = 3,4,... \end{aligned} \tag{5.19}$$

The approximate solution to (5.1)-(5.2) is the summation of (5.19).

Since the solution to (5.1)-(5.2) is expressed as an infinite series, then for practical purposes we have to terminate this infinite series. The $n+1$ term approximate solution to (5.1)-(5.2) will be given by

$$\phi(\bar{x}, t) = \sum_{j=0}^{n} y_j(\bar{x}, t), \tag{5.20}$$

with $y_j(\bar{x}, t)$ terms taken from (5.10), (5.15)or (5.19).

5.3 Uniqueness and Existence of fractional differential equations

This section will focus on the uniqueness and existence theorems of some of the fractional differential equations.

Whilst it is true that the theory concerning the uniqueness and existence of ordinary differential equations is well established, the same does not always hold true for partial differential equations. This is also applicable to fractional differential equations. The notorious question of proving the existence and uniqueness of Navier-Stokes non linear partial differential equations in fluid mechanics still remains as one of the open questions in mathematics. However, there is some noticeable progress in the field regarding proof of uniqueness and existence of fractional partial differential equations see [15].

In this section we are going to limit ourselves to the uniqueness and existence theory of fractional ordinary differential equations, for a discussion regarding the uniqueness and existence of fractional partial differential equations see [15].

We want to consider the fractional ordinary differential equation version of (5.1)-(5.2), thus we have,

$$\mathcal{D}_t^{\mu} y(t) + Ry(t) + Gy(t) = f(t), \quad p-1 < \mu \leq p, \quad p \in \mathbb{N}, \tag{5.21}$$

with initial conditions,

$$y^{(m)}(0) = \frac{d^m y(0)}{dt^m} = y_m(\bar{x}), \quad m = 0, 1, 2, ..., p-1. \tag{5.22}$$

If we take the Riemann-Liouville's fractional integral on both sides of (5.21) and utilise the initial conditions (5.22), then in view of the third property of the Riemann-Liouville fractional integral that was discussed in chapter 2 we have,

$$\begin{aligned} y(t) &= I_t^\mu f(t) + \sum_{m=0}^{p-1} \frac{t^m}{m!} y^{(m)}(0) - I_t^\mu R(y(t)) - I_t^\mu G(y(t)) \\ &= \mathcal{K}(t) - I_t^\mu R(y(t)) - I_t^\mu G(y(t)). \end{aligned} \tag{5.23}$$

I_t^μ is the Riemann-Liouville fractional integral and $K(t)$ is the term that arises due to the known function $f(t)$ and the initial conditions.

The following theorem and lemmas will form the basis of our proof of the uniqueness and existence theorem of the fractional differential equations.

We start with the Banach fixed point theorem, we will state this theorem without proof. The Banach fixed point theorem guarantees the existence of a unique solution of a differential equation.

Theorem 5.3.1 *We consider a set of continuous functions $X = C(T)$ on the interval $I = [0, T]$, where X is the Banach space. Then the mapping $\eta : X \to X$ is a contraction if for y and $\bar{y} \in X$ there exists $0 \leq L < 1$ such that $\| \eta(y) - \eta(\bar{y}) \| \leq L \| y - \bar{y} \|$. L is called the Lipschitz constant.*

We state the following lemmas with the assumptions that R and G operators are Lipschitzian.

Lemma 5.3.1 *Let $y, \bar{y} \in X$, then $\| R(y) - R(\bar{y}) \| \leq L_R \| y - \bar{y} \|$ and*
$\| G(y) - G(\bar{y}) \| \leq L_G \| y - \bar{y} \|$, for $0 \leq L_R < 1$ and $0 \leq L_G < 1$.

In view of the Riemann-Liouville's fractional integral definition (2.7), we state the following lemma.

Lemma 5.3.2 *In definition (2.7) if $f(s) = 1$, then we have,*
$\frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)ds}{(t-s)^{1-\mu}} = \frac{t^\mu}{\Gamma(\mu+1)}, \quad \mu \in (p-1, p], \quad p \in \mathbb{N}.$

We now state and prove a theorem that guarantees the existence of a unique solution of (5.21) - (5.22) under certain conditions.

Theorem 5.3.2 (5.21)-(5.22) *possesses a unique solution if* $\gamma \in [0, 1)$*, where* $\gamma = \frac{t^{\mu}}{\Gamma(\mu+1)}(L_R + L_G)$.

Proof

The solution of (5.21)-(5.22) is (5.23), thus we define the mapping

$$\eta(y(t)) = \mathcal{K}(t) - I_t^{\mu} R(y(t)) - I_t^{\mu} G(y(t)), \tag{5.24}$$

where η operates in a Banach space $\eta : X \to X$. Let y and $\bar{y} \in X$ then, we use some of the properties of norms given in appendix B in this proof

$$\begin{aligned}
\| \eta(y) - \eta(\bar{y}) \| &= \| -I_t^{\mu} R(y) - I_t^{\mu} G(y) + I_t^{\mu} R(\bar{y}) + I_t^{\mu} G(\bar{y}) \| \\
&= \| -I_t^{\mu} [R(y) - R(\bar{y})] - I_t^{\mu} [G(y) - G(\bar{y})] \| \\
&= \| I_t^{\mu} [R(y) - R(\bar{y})] + I_t^{\mu} [G(y) - G(\bar{y})] \| \\
&\leq \| I_t^{\mu} [R(y) - R(\bar{y})] \| + \| I_t^{\mu} [G(y) - G(\bar{y})] \| \\
&\leq I_t^{\mu} \Big(\| [R(y) - R(\bar{y})] \| + \| [G(y) - G(\bar{y})] \| \Big) \\
&\leq I_t^{\mu} \Big(L_R \| y - \bar{y} \| + L_G \| y - \bar{y} \| \Big) \\
&= \Big(L_R + L_G \Big) I_t^{\mu} \| y - \bar{y} \| \\
&= \frac{t^{\mu}}{\Gamma(\mu + 1)} (L_R + L_G) \| y - \bar{y} \| \\
&= \gamma \| y - \bar{y} \| .
\end{aligned}$$

Thus

$$\| \eta(y) - \eta(\bar{y}) \| \leq \gamma \| y - \bar{y} \| .$$

Therefore the above mapping is a contraction if $\gamma \in [0, 1)$, thus according to the Banach fixed point theorem there exists exist a unique solution to (5.21) - (5.22).

5.4 Natural transform decomposition method for the Fractional Klein-Gordon differential equation

In this section we are going to solve a particular case of the time fractional Klein-Gordon differential equation(FKGDE). The general time fractional Klein-Gordon differential equation is given in full in [2].

The Klein-Gordon differential equation has important applications in mathematical physics and engineering, it has been used to model problems in quantum field theory and solitons [2]. The FKGDE has also been considered with the Caputo fractional derivative in the applications in mathematical physics [2].

Consider the following Klein-Gordon differential equation of fractional order [2],

$$\begin{aligned}&\mathcal{D}_t^{\mu} v(x,t) - v_{xx}(x,t) + v^2(x,t) = 0, \quad \mu \in (1,2] \quad t > 0, x \in \mathbb{R} \qquad (5.25)\\ &v(x,0) = sin(x) + 1, \quad v_t(x,0) = 0.\end{aligned}$$

We take the natural transform on both sides of (5.25) and utilise the initial conditions to get,

$$\psi(x,s,u) = \frac{sin(x)+1}{s} + \left(\frac{u}{s}\right)^{\mu} \mathcal{N}[v_{xx} - v^2(x,t)]. \tag{5.26}$$

We then take the inverse natural transform on both sides of (5.26) to get

$$v(x,t) = sin(x) + 1 + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[v_{xx}(x,t) - v^2(x,t)]\right]. \tag{5.27}$$

We now apply the decomposition methods to (5.27), we will start with the ADM. In view of (5.7) and (5.8) we have

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) \tag{5.28}$$

and

$$v^2(x,t) = \sum_{n=0}^{\infty} A_n(x,t). \tag{5.29}$$

Substituting (5.28) and (5.29) into (5.27) gives,

$$\sum_{n=0}^{\infty} v_n(x,t) = sin(x) + 1 + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}\left[\sum_{n=0}^{\infty} v_{nxx}(x,t) - \sum_{n=0}^{\infty} A_n(x,t)\right]\right]. \tag{5.30}$$

The following iteration is then deduced from (5.30),

$$\begin{aligned}
v_0(x,t) &= sin(x)+1,\\
v_1(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[v_{0xx}(x,t)-A_0]\right]\\
&= \frac{-t^{\mu}}{\Gamma(\mu+1)}\left(sin^2(x)+3sin(x)+1\right),\\
v_2(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[v_{1xx}(x,t)-A_1]\right]\\
&= \frac{t^{2\mu}}{\Gamma(2\mu+1)}\left(2sin^3(x)+12sin^2(x)+11sin(x)\right),\\
v_3(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[v_{2xx}(x,t)-A_2]\right]\\
&= \frac{-t^{3\mu}}{\Gamma(3\mu+1)}\left(4sin^4(x)+46sin^3(x)+94sin^2(x)+21sin(x)-24\right)\\
&- \frac{\Gamma(2\mu+1)t^{3\mu}}{\Gamma(\mu+1)^2\Gamma(3\mu+1)}\left(sin^4(x)+6sin^3(x)+11sin^2(x)+6sin(x)+1\right).
\end{aligned} \tag{5.31}$$

Then the approximate solution to (5.25) is $v(x,t) = v_0 + v_1 + v_2 + v_3$.

The second decomposition method we will apply to (5.27) is the HPM. We first construct the homotopy of (5.27) as,

$$v(x,t) = sin(x)+1+q\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}[v_{xx}(x,t)-v^2(x,t)]\right], \quad q\in[0,1]. \tag{5.32}$$

In view of (5.12) and (5.13), we have

$$v(x,t) = \sum_{n=0}^{\infty} q^n v_n(x,t), \tag{5.33}$$

and the non linear term is written as

$$v^2(x,t) = \sum_{n=0}^{\infty} q^n A_n(x,t). \tag{5.34}$$

Substituting (5.33) and (5.34) into (5.32) yields

$$\sum_{n=0}^{\infty} q^n v_n(x,t) = sin(x)+1+q\mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathcal{N}\left[\sum_{n=0}^{\infty} q^n v_{nxx}(x,t)-\sum_{n=0}^{\infty} q^n A_n(x,t)\right]\right]. \tag{5.35}$$

We compare the terms with the similar powers of q in (5.35), this gives,

$$
\begin{aligned}
q^0 : v_0(x,t) &= \sin(x) + 1,\\
q^1 : v_1(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}\left[v_{0xx}(x,t) - A_0(x,t)\right]\right]\\
&= \frac{-t^{\mu}}{\Gamma(\mu+1)}\left(sin^2(x) + 3sin(x) + 1\right),\\
q^2 : v_2(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}\left[v_{1xx}(x,t) - A_1(x,t)\right]\right]\\
&= \frac{t^{2\mu}}{\Gamma(2\mu+1)}\left(2sin^3(x) + 12sin^2(x) + 11sin(x)\right),\\
q^3 : v_3(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}\left[v_{2xx}(x,t) - A_2(x,t)\right]\right]\\
&= \frac{-t^{3\mu}}{\Gamma(3\mu+1)}\left(4sin^4(x) + 46sin^3(x) + 94sin^2(x) + 21sin(x) - 24\right)\\
&- \frac{\Gamma(2\mu+1)t^{3\mu}}{\Gamma(\mu+1)^2\Gamma(3\mu+1)}\left(sin^4(x) + 6sin^3(x) + 11sin^2(x) + 6sin(x) + 1\right).
\end{aligned}
$$

The approximate solution to (5.25) is then the sum of the terms from above. We note that this is the same solution that we got when we combined the natural transform and the ADM.

We now apply our proposed method, we decompose (5.27) using the DJM. We write the non linear term in (5.27) as $G(v) = v^2(x,t)$, thus (5.27) becomes

$$
v(x,t) = sin(x) + 1 + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[v_{xx}(x,t) - G(v)]\right]. \tag{5.36}
$$

We note that in view of (5.7), we get (5.28), thus we substitute (5.28) into (5.36), this gives

$$
\sum_{n=0}^{\infty} v_n(x,t) = sin(x) + 1 + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}\left[\sum_{n=0}^{\infty} v_{nxx}(x,t) - G\left(\sum_{n=0}^{\infty} v_n\right)\right]\right]. \tag{5.37}
$$

In view of (5.17), $G\left(\sum\limits_{n=0}^{\infty} v_n\right)$ is then written as,

$$
G\left(\sum_{n=0}^{\infty} v_n\right) = G(v_0) + \sum_{n=1}^{\infty}\left[G\left(\sum_{k=0}^{n} v_k\right) - G\left(\sum_{k=0}^{n-1} v_k\right)\right]. \tag{5.38}
$$

Substituting (5.38) into (5.37), we get

$$\begin{aligned}\sum_{n=0}^{\infty} v_n(x,t) &= sin(x) + 1 + \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}\left[\sum_{n=0}^{\infty} v_{nxx}(x,t)\right.\right.\\ &- \left.\left.\left\{G(v_0) + \sum_{n=1}^{\infty}\left[G\left(\sum_{k=0}^{n} v_k\right) - G\left(\sum_{k=0}^{n-1} v_k\right)\right]\right\}\right]\right]. \qquad (5.39)\end{aligned}$$

The following iteration is then deduced from (5.39),

$$\begin{aligned}
v_0(x,t) &= sin(x) + 1,\\
v_1(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}[v_{0xx}(x,t) - G(v_0)]\right]\\
&= \frac{-t^{\mu}}{\Gamma(\mu+1)}\left(sin^2(x) + 3sin(x) + 1\right),\\
v_2(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}\left[v_{1xx}(x,t) - \{G(v_0+v_1) - G(v_0)\}\right]\right]\\
&= \frac{t^{2\mu}}{\Gamma(2\mu+1)}\left(2sin^3(x) + 12sin^2(x) + 11sin(x)\right)\\
&- \frac{\Gamma(2\mu+1)t^{3\mu}}{\Gamma(\mu+1)^2\Gamma(3\mu+1)}\left(sin^4(x) + 6sin^3(x) + 11sin^2(x) + 6sin(x) + 1\right),\\
v_3(x,t) &= \mathcal{N}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathcal{N}\left[v_{2xx}(x,t) - \{G(v_0+v_1+v_2) - G(v_0+v_1)\}\right]\right]\\
&= \frac{-t^{3\mu}}{\Gamma(3\mu+1)}\left(4sin^4(x) + 46sin^3(x) + 94sin^2(x) + 21sin(x) - 24)\right)\\
&+ \frac{\Gamma(2\mu+1)t^{4\mu}}{\Gamma(\mu+1)^2\Gamma(4\mu+1)}\left(2sin^5(x) + 30sin^4(x) + 88sin^3(x) + 66sin^2(x) - 16sin(x) - 20\right)\\
&+ \frac{\Gamma(3\mu+1)t^{4\mu}}{\Gamma(\mu+1)\Gamma(2\mu+1)\Gamma(4\mu+1)}\left(4sin^5(x) + 36sin^4(x) + 98sin^3(x) + 90\sin^2(x) + 22\sin(x)\right)\\
&- \frac{\Gamma(2\mu+1)\Gamma(4\mu+1)t^{5\mu}}{\Gamma(\mu+1)^3\Gamma(3\mu+1)\Gamma(5\mu+1)}\Big(2\sin^6(x) + 18\sin^5(x) + 60\sin^4(x) + 90\sin^3(x) + 60\sin^2 x\\
&+ 18\sin(x) + 2\Big)\\
&- \frac{\Gamma(4\mu+1)t^{5\mu}}{\Gamma(2\mu+1)^2\Gamma(5\mu+1)}\left(4\sin^6(x) + 48\sin^5(x) + 188\sin^4(x) + 264\sin^3(x) + 121\sin^2(x)\right)\\
&+ \frac{\Gamma(5\mu+1)t^{6\mu}}{\Gamma(\mu+1)^2\Gamma(3\mu+1)\Gamma(6\mu+1)}\Big(4\sin^7(x) + 48\sin^6(x) + 210\sin^5(x) + 420\sin^4(x) + 390\sin^3(x)\\
&+ 156\sin^2(x) + 22\sin(x)\Big)\\
&- \frac{\Gamma(2\mu+1)^2\Gamma(6\mu+1)t^{7\mu}}{\Gamma(\mu+1)^4\Gamma(3\mu+1)^2\Gamma(7\mu+1)}\Big(\sin^8(x) + 12\sin^7(x) + 58\sin^6(x)\\
&+ 144\sin^5(x) + 195\sin^4(x) + 144\sin^3(x) + 58\sin^2(x) + 12\sin(x) + 1\Big).
\end{aligned}$$

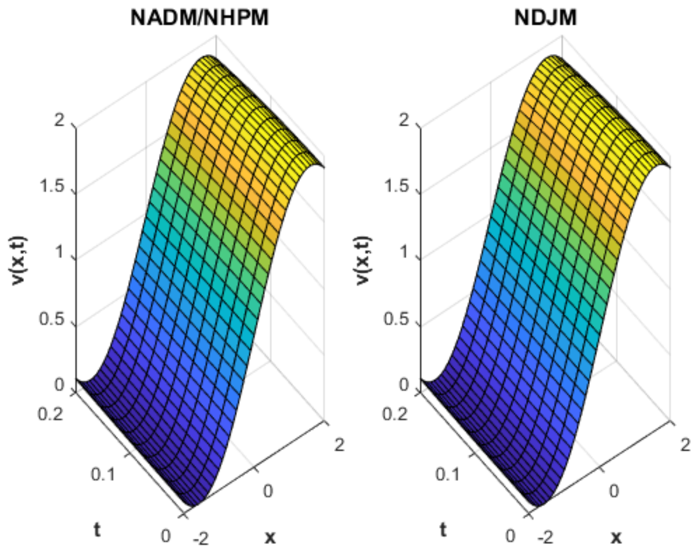

Figure 5.1: 3D plots for the approximate solution $v(x,t)$ of (5.25) for $\mu = 2$, $-2 \leq x \leq 2$ and $0 \leq t \leq 0.2$ using the NADM, NHPM and NDJM.

Then the approximate solution to (5.25) from the above iteration is $v(x,t) = v_0 + v_1 + v_2 + v_3$.

We note that the solutions from the NADM and the NHPM are exactly the same, but our proposed method the NDJM gives a different solution. The NDJM gives more terms in it's approximate solution than the NADM and the NHPM.

The approximate solutions to (5.25) using the NADM, NHPM and the NDJM are compared in the 3D plots from Figures 5.1 - 5.4 with different values of μ. The 3D plots were generated using MATLAB.

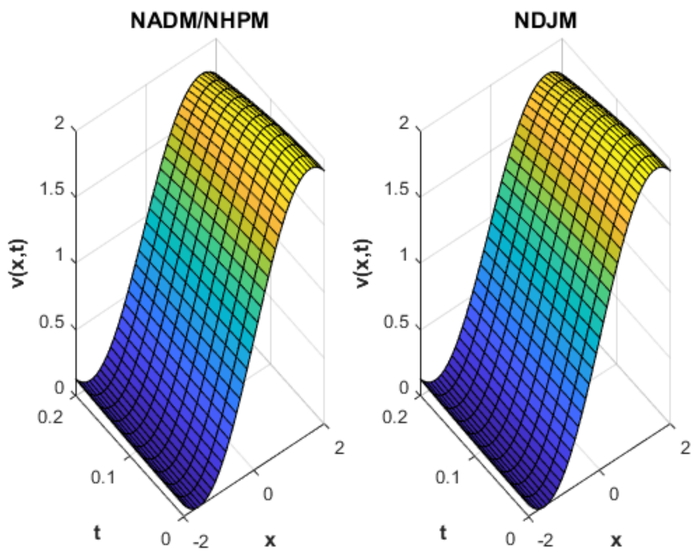

Figure 5.2: 3D plots for the approximate solution $v(x,t)$ of (5.25) for $\mu = 1.7$, $-2 \le x \le 2$ and $0 \le t \le 0.2$ using the NADM, NHPM and NDJM.

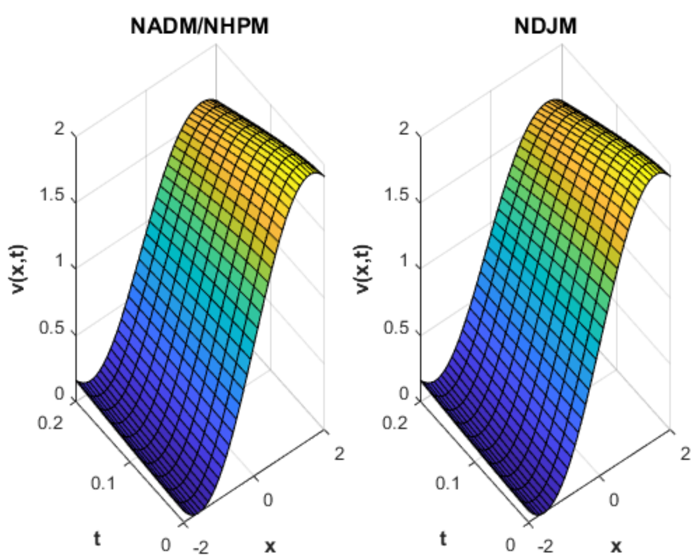

Figure 5.3: 3D plots for the approximate solution $v(x,t)$ of (5.25) for $\mu = 1.4$, $-2 \le x \le 2$ and $0 \le t \le 0.2$ using the NADM, NHPM and NDJM.

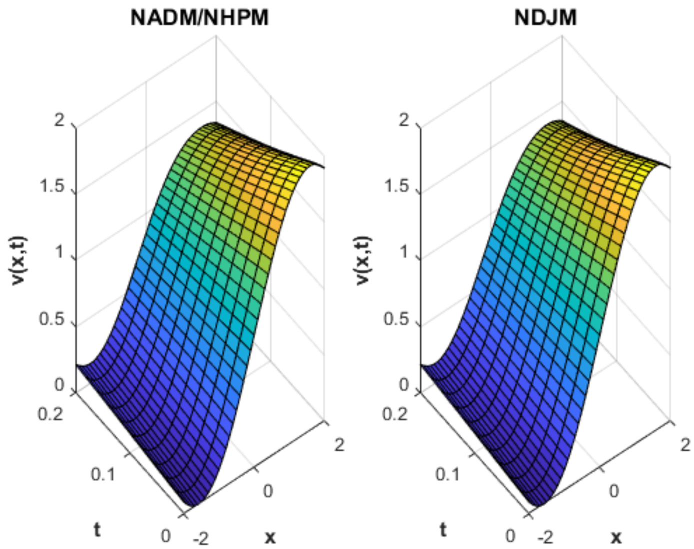

Figure 5.4: 3D plots for the approximate solution $v(x,t)$ of (5.25) for $\mu = 1.1$, $-2 \leq x \leq 2$ and $0 \leq t \leq 0.2$ using the NADM, NHPM and NDJM.

We note from the 3D plots that for the same values of μ, all the three forms of the natural transform decomposition method give results that are in close agreement. In the diagrams from Figures 5.5 to 5.7, we hold x constant and observe how solutions change with varying t for different values of μ.

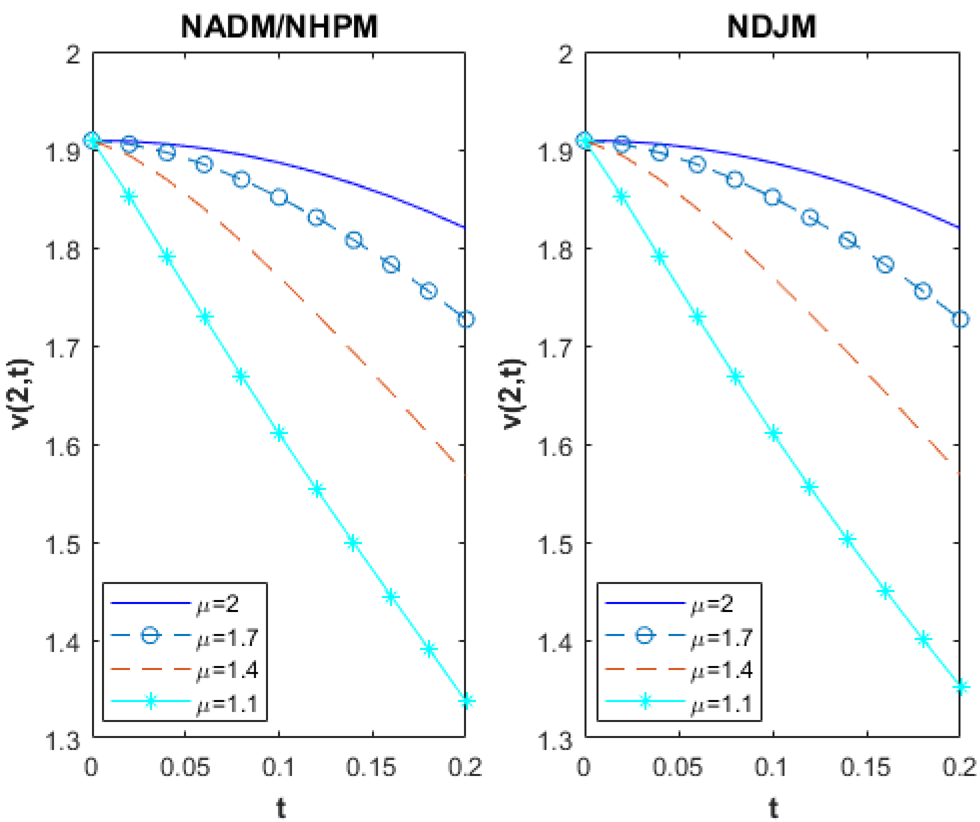

Figure 5.5: 2D plots for the approximate solution $v(x,t)$ of (5.25) at different values of μ, with $x = 2$ and $0 \leq t \leq 0.2$ using the NADM, NHPM and NDJM.

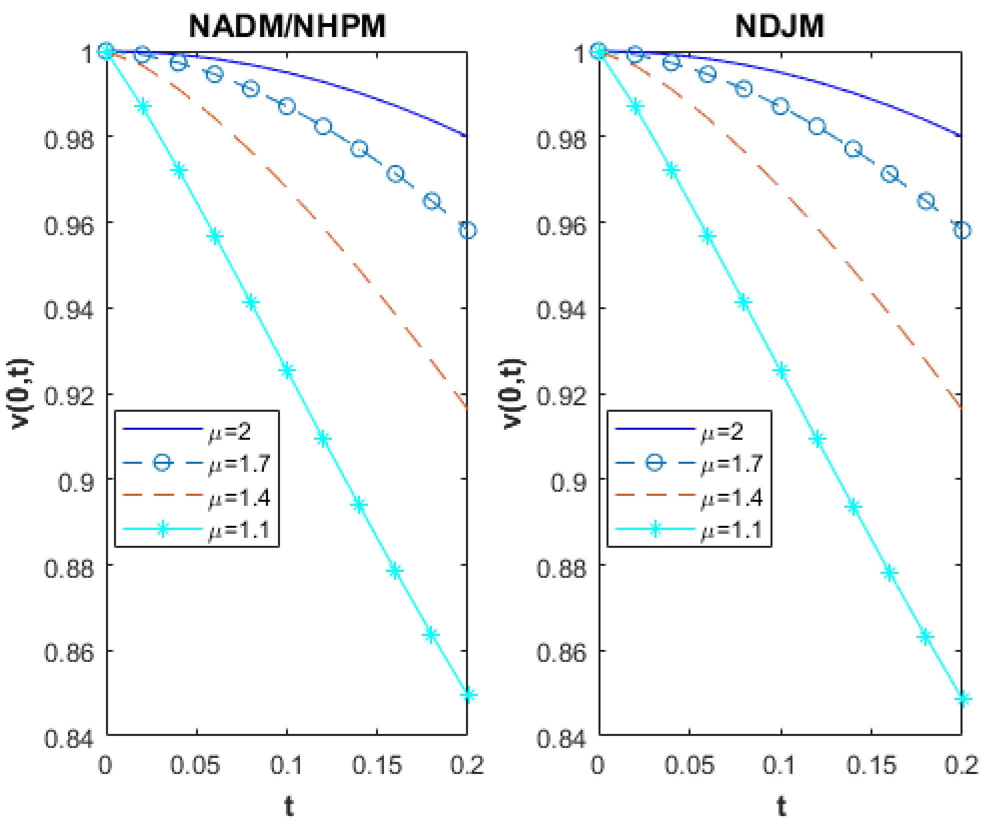

Figure 5.6: 2D plots for the approximate solution $v(x,t)$ of (5.25) at different values of μ, with $x = 0$ and $0 \leq t \leq 0.2$ using the NADM, NHPM and NDJM.

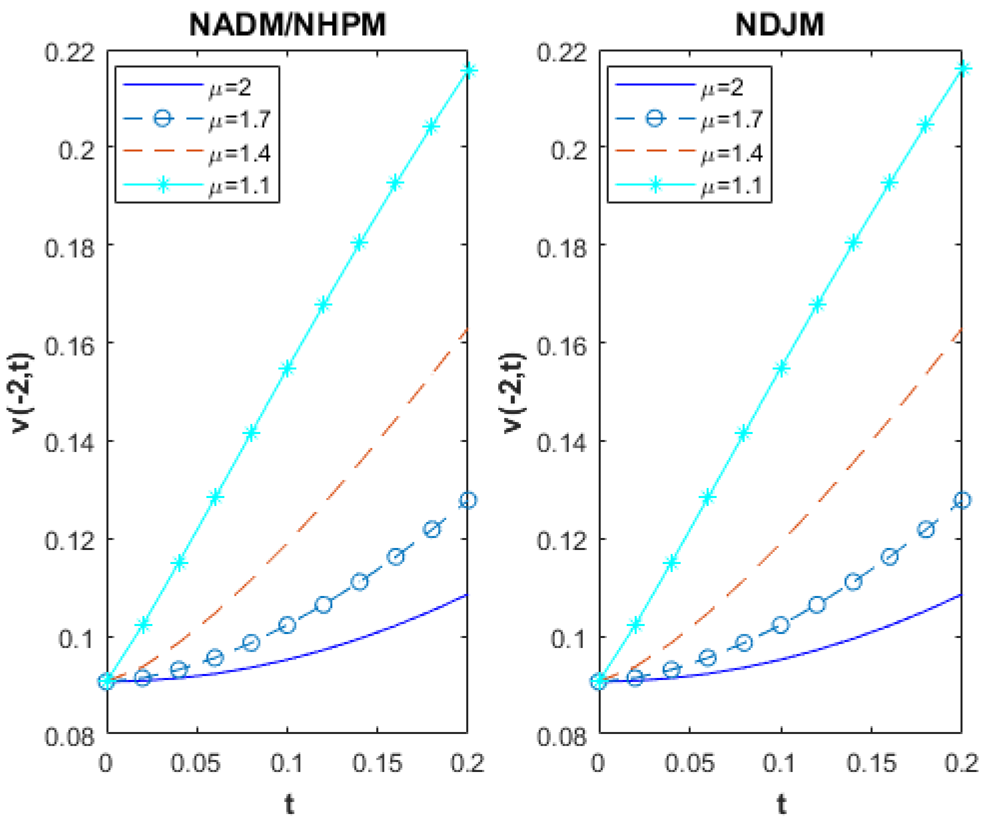

Figure 5.7: 2D plots for the approximate solution $v(x,t)$ of (5.25) at different values of μ, with $x = -2$ and $0 \leq t \leq 0.2$ using the NADM, NHPM and NDJM.

It is clear from Figure 5.5 to 5.7 that the results of the NADM/NHPM and the NDJM are in close agreement.

The Fractional reduced differential transform method (FRDTM) was used to solve (5.25) in [2]. We do a comparison of our results from the NTDM and the results from the FRDTM. These comparisons are shown from Tables 5.1 to 5.4 for different values of μ with the values of x held constant.

Table 5.1: Numerical results of example (5.25) when $\mu = 2$ and $x = 2$.

t	NADM/NHPM	NDJM	FRDTM
0.000	1.90929742682568	1.90929742682568	1.90929742682568
0.02	1.908386626838561	1.908386626838564	1.908386626838291
0.04	1.905655939848944	1.905655939849526	1.905655939831552
0.06	1.9011104972646221	1.901110497279505	1.901110497066512
0.08	1.894758826407097	1.894758826555686	1.894758825293991
0.10	1.886612812974378	1.886612813859408	1.886612808728212
0.12	1.876687648488873	1.876687652291021	1.876687635809891
0.14	1.865001762730399	1.865001775766730	1.865001730758741
0.16	1.851576741154314	1.851576779048659	1.851576669915473
0.18	1.836437227294754	1.836437324392558	1.836437082873210
0.20	1.819610810152987	1.819616219619335	1.819610538398373

Table 5.2: Numerical results of example (5.25) when $\mu = 1.7$ and $x = 2$.

t	NADM/NHPM	NDJM	FRDTM
0.000	1.90929742682568	1.90929742682568	1.90929742682568
0.02	1.905487037373375	1.905487037375226	1.905487036774822
0.04	1.896943184920641	1.896943185126714	1.896943164392221
0.06	1.884757245633752	1.884757248875763	1.884757083295460
0.08	1.869427974573217	1.869427997460931	1.869427270515072
0.10	1.851294978232886	1.851295082378849	1.851292781134051
0.12	1.830623618000142	1.830623976909348	1.830618050324223
0.14	1.807636976895555	1.807637997830932	1.807624756012867
0.16	1.782530950946607	1.782533474376956	1.782506804039867
0.18	1.755482305237536	1.755487907380187	1.755438276182028
0.20	1.726653296710266	1.726664723215268	1.726577943359134

Table 5.3: Numerical results of example (5.25) when $\mu = 1.4$ and $x = 2$.

t	NADM/NHPM	NDJM	FRDTM
0.000	1.90929742682568	1.90929742682568	1.90929742682568
0.02	1.894040532260641	1.894040533517336	1.894040387584075
0.04	1.869373519986955	1.869373580697951	1.869370860951241
0.06	1.839589954035192	1.839590539164776	1.839575355561266
0.08	1.806230572386507	1.806233486314197	1.806181701507295
0.10	1.770220932788784	1.770231036765096	1.770096173696317
0.12	1.732206596804765	1.732234458451996	1.731938288917010
0.14	1.692670528356695	1.692736114998361	1.692157891346995
0.16	1.651986613828648	1.652124114054211	1.651088407451716
0.18	1.610447998546299	1.610711833847320	1.608974951124361
0.20	1.568283637327962	1.568755706450438	1.565990668299373

Table 5.4: Numerical results of example (5.25) when $\mu = 1.1$ and $x = 2$.

t	NADM/NHPM	NDJM	FRDTM
0.000	1.90929742682568	1.90929742682568	1.90929742682568
0.02	1.852005365407891	1.852006047310028	1.851982564774923
0.04	1.790116984624319	1.790131177705032	1.789892417649126
0.06	1.728686738860605	1.728769992314518	1.727830791525423
0.08	1.668732261223284	1.669022977628821	1.666520466207839
0.10	1.610519133663055	1.611283110183686	1.605900134134536
0.12	1.553977919882964	1.555655136512741	1.545547561644351
0.14	1.498845324401937	1.502097673032489	1.484824600550028
0.16	1.444728477566549	1.450487558712816	1.422944164952713
0.18	1.391139592033295	1.400653974093608	1.359006894592961
0.20	1.337516853089243	1.35239837326527	1.292023609371004

We note from table 5.1 to 5.4 that as $\mu \to 2$, the results from the NTDM(NADM/NHPM and NDJM) and the FRDTM are in close agreement. However as the value of $\mu \to 1$, the methods tend to give different results. We also note that as we increase the value of t, the discrepancies in the results of the methods also increases, this result is consistent for all values of μ that we have used.

We now investigate the convergence and rates of convergence of the solutions of the NTDM when applied to (5.25). In view of (4.17), (4.18) and (4.19) in chapter 4, we increase the one dimensional case to two dimensional such that we have,

$$\| v_n(x,t) \| = \sqrt{\int_{-2}^{2}\int_{0}^{0.2} | v_n(x,t) |^2 \, dtdx}. \tag{5.40}$$

$$\| v_{n-1}(x,t) \| = \sqrt{\int_{-2}^{2}\int_{0}^{0.2} | v_{n-1}(x,t) |^2 \, dtdx}. \tag{5.41}$$

and

$$\rho_n = \frac{\| v_n(x,t) \|}{\| v_{n-1}(x,t) \|}, \quad n = 1, 2, ... \tag{5.42}$$

We assign the values of ρ_n and $\hat{\rho}_n$ to the NADM/NHPM and NDJM respectively. We then compute the numerical values of ρ_n and $\hat{\rho}_n$ for different values of μ with the help of MATHEMATICA.

When $\mu = 2$,

$$\begin{aligned}
\rho_1 &= \frac{\| v_1(x,t) \|}{\| v_0(x,t) \|} = 0.0200659 < 1, \\
\rho_2 &= \frac{\| v_2(x,t) \|}{\| v_1(x,t) \|} = 0.0112599 < 1, \\
\rho_3 &= \frac{\| v_3(x,t) \|}{\| v_2(x,t) \|} = 0.0121809 < 1.
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_1 &= \frac{\| v_1(x,t) \|}{\| v_0(x,t) \|} = 0.0200659 < 1, \\
\hat{\rho}_2 &= \frac{\| v_2(x,t) \|}{\| v_1(x,t) \|} = 0.0111891 < 1, \\
\hat{\rho}_3 &= \frac{\| v_3(x,t) \|}{\| v_2(x,t) \|} = 0.0057263 < 1.
\end{aligned}$$

When $\mu = 1.7$,

$$\begin{aligned}
\rho_1 &= \frac{\| v_1(x,t) \|}{\| v_0(x,t) \|} = 0.0448848 < 1, \\
\rho_2 &= \frac{\| v_2(x,t) \|}{\| v_1(x,t) \|} = 0.0336272 < 1, \\
\rho_3 &= \frac{\| v_3(x,t) \|}{\| v_2(x,t) \|} = 0.0357686 < 1.
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_1 &= \frac{\| v_1(x,t) \|}{\| v_0(x,t) \|} = 0.0448848 < 1, \\
\hat{\rho}_2 &= \frac{\| v_2(x,t) \|}{\| v_1(x,t) \|} = 0.0331074 < 1, \\
\hat{\rho}_3 &= \frac{\| v_3(x,t) \|}{\| v_2(x,t) \|} = 0.0195029 < 1.
\end{aligned}$$

When $\mu = 1.4$,

$$\begin{aligned}
\rho_1 &= \frac{\| v_1(x,t) \|}{\| v_0(x,t) \|} = 0.0973383 < 1, \\
\rho_2 &= \frac{\| v_2(x,t) \|}{\| v_1(x,t) \|} = 0.0956038 < 1, \\
\rho_3 &= \frac{\| v_3(x,t) \|}{\| v_2(x,t) \|} = 0.10317 < 1.
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_1 &= \frac{\| v_1(x,t) \|}{\| v_0(x,t) \|} = 0.0973383 < 1,\\
\hat{\rho}_2 &= \frac{\| v_2(x,t) \|}{\| v_1(x,t) \|} = 0.0921153 < 1,\\
\hat{\rho}_3 &= \frac{\| v_3(x,t) \|}{\| v_2(x,t) \|} = 0.0617062 < 1.
\end{aligned}$$

When $\mu = 1.1$,

$$\begin{aligned}
\rho_1 &= \frac{\| v_1(x,t) \|}{\| v_0(x,t) \|} = 0.204051 < 1,\\
\rho_2 &= \frac{\| v_2(x,t) \|}{\| v_1(x,t) \|} = 0.256454 < 1,\\
\rho_3 &= \frac{\| v_3(x,t) \|}{\| v_2(x,t) \|} = 0.288323 < 1.
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_1 &= \frac{\| v_1(x,t) \|}{\| v_0(x,t) \|} = 0.204051 < 1,\\
\hat{\rho}_2 &= \frac{\| v_2(x,t) \|}{\| v_1(x,t) \|} = 0.235372 < 1,\\
\hat{\rho}_3 &= \frac{\| v_3(x,t) \|}{\| v_2(x,t) \|} = 0.17642 < 1.
\end{aligned}$$

We note two important observations. Firstly, the NTDM(NADM/NHPM and NDJM) solutions converge because all the values of ρ_n and $\hat{\rho}$ are less than 1. Secondly, comparing the values of ρ_n and $\hat{\rho}$ from above, we note that the values of $\hat{\rho}$ are lesser than the values of ρ_n for the same value of μ, that's in this case our proposed technique (NDJM) has a higher rate of convergence than the NADM/NHPM.

Chapter 6

A modified form of the Caputo fractional derivative

6.1 Introduction

In this chapter we still use the Natural transform decomposition method (NTDM) as we did in the previous chapter, but this time the fractional derivative is considered in the Caputo-Fabrizio sense. The definition of the Caputo-Fabrizio fractional derivative was given in chapter 2.

We will describe the methodology in detail, use this method to solve Klein-Gordon differential equation of fractional order and then compare our results with the ones in the previous chapter.

6.2 General description of the method

Consider the following type of the time fractional differential equation,

$$\mathbb{D}_t^{\mu} y(\bar{x},t) + Ry(\bar{x},t) + Gy(\bar{x},t) = f(\bar{x},t), \quad p-1 < \mu \leq p, \quad p \in \mathbb{N}, \tag{6.1}$$

with initial conditions,

$$y^{(m)}(\bar{x},0)=\frac{\partial^m y(\bar{x},0)}{\partial t^m}=y_m(\bar{x}),\quad m=0,1,2,...,p-1. \tag{6.2}$$

$\bar{x}=(x_1,x_2,...)$, $\mathbb{D}_t^{\mu}$ is the Caputo-Fabrizio fractional derivative of order μ, R represents a linear operator, $Gy(\bar{x},t)$ is the non linear term and $f(\bar{x},t)$ is taken as the source term.

In the first step of the method we take the natural transform on both sides of (6.1),

$$\mathcal{N}[\mathbb{D}_t^{\mu}y(\bar{x},t)]+\mathcal{N}[Ry(\bar{x},t)]+\mathcal{N}[Gy(\bar{x},t)]=\mathcal{N}[f(\bar{x},t)], \tag{6.3}$$

We remind that from chapter 3,

$$\mathcal{N}[\mathbb{D}_t^{\mu}y(\bar{x},t)]=\mathcal{N}[\mathbb{D}_t^{\alpha+p}y(\bar{x},t)],\quad p\in\mathbb{N}\cup 0,\quad \alpha\in(0;1]. \tag{6.4}$$

We use (3.20) and the initial conditions (6.2) in (6.3), this yields

$$\begin{aligned}\psi(\bar{x},s,u) &= \tfrac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}[f(\bar{x},t)]+\tfrac{u^p}{s^{p+1}}\sum_{m=0}^{p}\left(\tfrac{s}{u}\right)^{p-m}y^{(m)}(0)\\ &\quad -\tfrac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}[Ry(\bar{x},t)+Gy(\bar{x},t)].\end{aligned} \tag{6.5}$$

In the second step, we take the inverse natural transform on both sides of (6.5), this gives

$$y(\bar{x},t)=\mathcal{K}(x,t)-\mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}[Ry(\bar{x},t)+Gy(\bar{x},t)]\right], \tag{6.6}$$

where $\mathcal{K}(x,t)$ is the term due to the source term and the initial conditions.

The third is to apply the decomposition methods to (6.6). We will use three decomposition methods as we did in the previous chapter, the ADM, HPM and DJM.

We start with the ADM, we substitute (5.7) and (5.8) into (6.6), this yields

$$\sum_{n=0}^{\infty}y_n(\bar{x},t)=\mathcal{K}(x,t)-\mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}\left[R\sum_{n=0}^{\infty}y_n(\bar{x},t)+\sum_{n=0}^{\infty}A_n(\bar{x},t)\right]\right]. \tag{6.7}$$

The following iteration is then deduced from (6.7)

$$\begin{aligned}y_0(\bar{x},t) &= \mathcal{K}(x,t),\\ y_n(\bar{x},t) &= -\mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}[Ry_{n-1}(\bar{x},t)+A_{n-1}(\bar{x},t)]\right],\quad n=1,2,...\end{aligned} \tag{6.8}$$

Then the solution to (6.1)-(6.2) is the summation of the terms from the above iteration.

Applying HPM to (6.6), we construct the homotopy of (6.6) as

$$y(\bar{x},t) = \mathcal{K}(x,t) - q\mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}[Ry(\bar{x},t) + Gy(\bar{x},t)]\right], \quad q \in [0,1]. \tag{6.9}$$

Substituting (5.12) and (5.13) into (6.9) yields,

$$\sum_{n=0}^{\infty} q^n y_n(\bar{x},t) = \mathcal{K}(x,t) - q\mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}\left[R\sum_{n=0}^{\infty} q^n y_n(\bar{x},t) + \sum_{n=0}^{\infty} q^n A_n(\bar{x},t)\right]\right]. \tag{6.10}$$

We then compare terms with the same powers of q from (6.10)

$$\begin{aligned} q^0 : y_0(\bar{x},t) &= \mathcal{K}(x,t), \\ q^n : y_n(\bar{x},t) &= -\mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}[Ry_{n-1}(\bar{x},t) + A_{n-1}(\bar{x},t)]\right], \quad n = 1,2,... \end{aligned}$$

The solution to (6.1)-(6.2) is the sum of the terms obtained from the above iteration.

We then consider our proposed method, we apply the DJM to (6.6), we substitute (5.7) into (6.6) to get

$$\sum_{n=0}^{\infty} y_n(\bar{x},t) = \mathcal{K}(\bar{x},t) - \mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}\left[R\sum_{n=0}^{\infty} y_n(\bar{x},t) + G\sum_{n=0}^{\infty} y_n(\bar{x},t)\right]\right]. \tag{6.11}$$

We then substitute (5.17) into (6.11), this gives

$$\begin{aligned} \sum_{n=0}^{\infty} y_n(\bar{x},t) &= \mathcal{K}(\bar{x},t) - \mathcal{N}^{-1}\Bigg[\tfrac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}\Bigg[R\sum_{n=0}^{\infty} y_n(\bar{x},t) + G(y_0(\bar{x},t)) \\ &\quad + \sum_{n=1}^{\infty}\left[G\left(\sum_{k=0}^{n} y_k(\bar{x},t)\right) - G\left(\sum_{k=0}^{n-1} y_k(\bar{x},t)\right)\right]\Bigg]\Bigg]. \end{aligned} \tag{6.12}$$

The following iteration is then deduced from (6.12)

$$\begin{aligned} y_0(\bar{x},t) &= \mathcal{K}(x,t), \\ y_1(\bar{x},t) &= -\mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}\left[Ry_0(\bar{x},t) + G(y_0)\right]\right], \\ y_n(\bar{x},t) &= -\mathcal{N}^{-1}\left[\frac{u^p(s-\alpha(s-u))}{s^{p+1}}\mathcal{N}\left[R\sum_{n=0}^{\infty} y_{n-1}(\bar{x},t) + G\left(\sum_{k=0}^{n-1} y_k\right) - G\left(\sum_{k=0}^{n-2} y_k\right)\right]\right], \quad n = 2,3,... \end{aligned}$$

The solution to (6.1)-(6.2) is the summation of the terms from the above iteration.

6.3 Solution of the Fractional Klein-Gordon equation

We now apply the methodology that we described in the previous section to the Fractional Klein-Gordon differential equation (5.25).

Consider

$$\mathbb{D}_t^{\mu} v(x,t) - v_{xx}(x,t) + v^2(x,t) = 0, \quad \mu \in (1,2] \quad t > 0, x \in \mathbb{R} \tag{6.13}$$
$$v(x,0) = sin(x) + 1, \quad v_t(x,0) = 0$$

Firstly, we note that for (6.13) $\mu = \alpha + 1$ thus in view of (3.20),

$$\begin{aligned} \mathcal{N}[\mathbb{D}_t^{\mu} v(x,t)] &= \mathcal{N}[\mathbb{D}_t^{\alpha+1} v(x,t)] \\ &= \frac{1}{s - \alpha(s-u)} \left[s\left(\frac{s}{u}\right)\psi(x,s,u) - \left(\frac{s}{u}\right) v(x,0) - v_t(x,0) \right], \alpha \in (0,1]. \end{aligned} \tag{6.14}$$

Taking the natural transform on both sides of (6.13), utilising (6.14) and the initial conditions, we thereafter solve for $\psi(x,s,u)$ to get,

$$\psi(x,s,u) = \frac{1}{s}\left(sin(x) + 1\right) + \frac{u(s - \alpha(s-u))}{s^2} \mathcal{N}[v_{xx}(x,t) - v^2(x,t)]. \tag{6.15}$$

We then take the inverse natural transform on both sides of (6.15), this yields

$$v(x,t) = sin(x) + 1 + \mathcal{N}^{-1}\left[\frac{u(s - \alpha(s-u))}{s^2} \mathcal{N}[v_{xx}(x,t) - v^2(x,t)] \right]. \tag{6.16}$$

The next stage of the method entails applying the decomposition methods to (6.16). We start by applying the ADM, thus, we substitute (5.28) and (5.29) into (6.16) to get,

$$\sum_{n=0}^{\infty} v_n(x,t) = sin(x) + 1 + \mathcal{N}^{-1}\left[\frac{u(s - \alpha(s-u))}{s^2} \mathcal{N}\left[\sum_{n=0}^{\infty} v_{nxx}(x,t) - \sum_{n=0}^{\infty} A_n(x,t) \right] \right]. \tag{6.17}$$

The following results are then deduced from (6.17)

$$
\begin{aligned}
v_0 &= sin(x) + 1,\\
v_1 &= \mathcal{N}^{-1}\left[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{0xx}(x,t) - A_0\right]\right]\\
&= \left(\frac{-\alpha t^2}{2} + \alpha t - t\right)(sin^2 x + 3 sinx + 1),\\
v_2 &= \mathcal{N}^{-1}\left[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{1xx}(x,t) - A_1\right]\right]\\
&= \left(\frac{\alpha^2 t^4}{24} - \frac{\alpha(\alpha-1)t^3}{3} + \frac{(\alpha-1)^2 t^2}{2}\right)\left(2 sin^3 x + 12 sin^2 x + 11 sinx\right),\\
v_3 &= \mathcal{N}^{-1}\left[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{2xx}(x,t) - A_2\right]\right]\\
&= \left(\frac{-\alpha^3 t^6}{120} + \frac{\alpha^2(\alpha-1)t^5}{10} - \frac{\alpha(\alpha-1)^2 t^4}{3} + \frac{(\alpha-1)^3 t^3}{3}\right)\left(sin^4 x + 6 sin^3 x + 11 sin^2 x + 6 sinx + 1\right)\\
&+ \left(\frac{-\alpha^3 t^6}{720} + \frac{\alpha^2(\alpha-1)t^5}{40} - \frac{\alpha(\alpha-1)^2 t^4}{8} + \frac{(\alpha-1)^3 t^3}{6}\right)\left(4 sin^4 x + 46 sin^3 x + 94 sin^2 x + 21 sinx - 24\right)
\end{aligned}
$$

The approximate solution to (6.13) is then given by $v(x,t) = v_0 + v_1 + v_2 + v_3$.

We then apply the HPM to (6.16), we construct the homotopy of (6.16) as

$$
v(x,t) = sin(x) + 1 + q\mathcal{N}^{-1}\left[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}[v_{xx}(x,t) - v^2(x,t)]\right], \quad q \in [0;1]. \quad (6.18)
$$

We then substitute (5.33) and (5.34) into (6.18), this gives

$$
\sum_{n=0}^{\infty} q^n v_n(x,t) = sin(x) + 1 + q\mathcal{N}^{-1}\left[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}\left[\sum_{n=0}^{\infty} q^n v_{nxx}(x,t) - \sum_{n=0}^{\infty} q^n A_n(x,t)\right]\right]. \quad (6.19)
$$

Comparing terms with the same powers of q in (6.19) we get

$$
\begin{aligned}
q^0 : v_0 &= sin(x) + 1, \\
q^1 : v_1 &= \mathcal{N}^{-1}\left[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{0xx}(x,t) - A_0\right]\right] \\
&= \left(\frac{-\alpha t^2}{2} + \alpha t - t\right)(sin^2 x + 3 sinx + 1), \\
q^2 : v_2 &= \mathcal{N}^{-1}\left[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{1xx}(x,t) - A_1\right]\right] \\
&= \left(\frac{\alpha^2 t^4}{24} - \frac{\alpha(\alpha-1)t^3}{3} + \frac{(\alpha-1)^2 t^2}{2}\right)\left(2sin^3 x + 12 sin^2 x + 11 sinx\right), \\
q^3 : v_3 &= \mathcal{N}^{-1}\left[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{2xx}(x,t) - A_2\right]\right] \\
&= \left(\frac{-\alpha^3 t^6}{120} + \frac{\alpha^2(\alpha-1)t^5}{10} - \frac{\alpha(\alpha-1)^2 t^4}{3} + \frac{(\alpha-1)^3 t^3}{3}\right)\left(sin^4 x + 6 sin^3 x + 11 sin^2 x + 6 sinx + 1\right) \\
&+ \left(\frac{-\alpha^3 t^6}{720} + \frac{\alpha^2(\alpha-1)t^5}{40} - \frac{\alpha(\alpha-1)^2 t^4}{8} + \frac{(\alpha-1)^3 t^3}{6}\right) \\
&\times \left(4 sin^4 x + 46 sin^3 x + 94 sin^2 x + 21 sinx - 24\right)
\end{aligned}
$$

The approximate solution to (6.13) is then the summation of the terms from above. We note that this gives the same solution as the NADM.

We now use our proposed method, we apply the DJM to (6.16), thus (6.16) becomes,

$$
\begin{aligned}
\sum_{n=0}^{\infty} v_n(x,t) &= sin(x) + 1 + \mathcal{N}^{-1}\Bigg[\frac{u(s-\alpha(s-u))}{s^2}\mathcal{N}\Bigg[v_{nxx}(x,t) \\
&- \left\{G(v_0) + \sum_{n=1}^{\infty}\left[G\left(\sum_{k=0}^{n} v_k\right) - G\left(\sum_{k=0}^{n-1} v_k\right)\right]\right\}\Bigg]\Bigg]. \qquad (6.20)
\end{aligned}
$$

We note that the non linear term in (6.16) is $G(v) = v^2$, then the following iteration is deduced from (6.20)

$$
\begin{aligned}
v_0 &= sinx + 1, \\
v_1 &= \mathcal{N}^{-1}\left[\frac{(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{0xx}(x,t) - G\left(v_0\right)\right]\right] \\
&= \left(\frac{-\alpha t^2}{2} + \alpha t - t\right)(sin^2 x + 3sinx + 1), \\
v_2 &= \mathcal{N}^{-1}\left[\frac{(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{1xx}(x,t) - \{G\left(v_0 + v_1\right) - G(v_0)\}\right]\right] \\
&= \left(\frac{\alpha^2 t^4}{24} - \frac{\alpha(\alpha-1)t^3}{3} + \frac{(\alpha-1)^2 t^2}{2}\right)\left(2sin^3 x + 12sin^2 x + 11sinx\right) \\
&+ \left(\frac{-\alpha^3 t^6}{120} + \frac{\alpha^2(\alpha-1)t^5}{10} - \frac{\alpha(\alpha-1)^2 t^4}{3} + \frac{(\alpha-1)^3 t^3}{3}\right)\left(sin^2 x + 3sinx + 1\right)^2, \\
v_3 &= \mathcal{N}^{-1}\left[\frac{(s-\alpha(s-u))}{s^2}\mathcal{N}\left[v_{2xx}(x,t) - \{G\left(v_0 + v_1 + v_2\right) - G(v_0 + v_1)\}\right]\right] \\
&= \left(\frac{-\alpha^3 t^6}{720} + \frac{\alpha^2(\alpha-1)t^5}{40} - \frac{\alpha(\alpha-1)^2 t^4}{8} + \frac{(\alpha-1)^3 t^3}{6}\right)\left(4sin^4 x + 46sin^3 x + 94sin^2 x + 21sinx - 24\right) \\
&+ \left(\frac{\alpha^4 t^8}{6720} - \frac{\alpha^3(\alpha-1)t^7}{280} + \frac{\alpha^2(\alpha-1)^2 t^6}{36} - \frac{\alpha(\alpha-1)^3 t^5}{12} + \frac{(\alpha-1)^4 t^4}{12}\right) \\
&\times \left(2sin^5 x + 30sin^4 x + 88sin^3 x + 66sin^2 x - 16sinx - 20\right) \\
&+ \left(\frac{\alpha^4 t^8}{2688} - \frac{\alpha^3(\alpha-1)t^7}{126} + \frac{13\alpha^2(\alpha-1)^2 t^6}{240} - \frac{17\alpha(\alpha-1)^3 t^5}{120} + \frac{(\alpha-1)^4 t^4}{8}\right) \\
&\times (2sin^2 x + 6sinx + 2)(2sin^3 x + 12sin^2 x + 11sinx) \\
&+ \left(\frac{-\alpha^5 t^{10}}{21600} + \frac{11\alpha^4(\alpha-1)t^9}{8640} - \frac{27\alpha^3(\alpha-1)^2 t^8}{2240} + \frac{\alpha^2(\alpha-1)^3 t^7}{20} - \frac{17\alpha(\alpha-1)^4 t^6}{180} + \frac{(\alpha-1)^5 t^5}{15}\right) \\
&\times (2sin^2 x + 6sinx + 2)(sin^2 x + 3sinx + 1)^2 \\
&+ \left(\frac{-\alpha^5 t^{10}}{51840} + \frac{\alpha^4(\alpha-1)t^9}{1728} - \frac{25\alpha^3(\alpha-1)^2 t^8}{4032} + \frac{5\alpha^2(\alpha-1)^3 t^7}{168} - \frac{23\alpha(\alpha-1)^4 t^6}{360} + \frac{1}{20}(\alpha-1)^5 t^5\right) \\
&\times (2sin^3 x + 12sin^2 x + 11sinx)^2 \\
&+ \left(\frac{\alpha^6 t^{12}}{380160} - \frac{\alpha^5(\alpha-1)t^{11}}{10560} + \frac{41\alpha^4(\alpha-1)^2 t^{10}}{32400} - \frac{211\alpha^3(\alpha-1)^3 t^9}{25920} + \frac{541\alpha^2(\alpha-1)^4 t^8}{20160}\right. \\
&- \left.\frac{11\alpha(\alpha-1)^5 t^7}{252} + \frac{(\alpha-1)^6 t^6}{36}\right)(4sin^3 x + 24sin^2 x + 22sinx)(sin^2 x + 3sinx + 1)^2 \\
&+ \left(\frac{-\alpha^7 t^{14}}{2620800} + \frac{\alpha^6(\alpha-1)t^{13}}{62400} - \frac{61\alpha^5(\alpha-1)^2 t^{12}}{237600} + \frac{41\alpha^4(\alpha-1)^3 t^{11}}{19800} - \frac{149\alpha^3(\alpha-1)^4 t^{10}}{16200}\right. \\
&+ \left.\frac{37\alpha^2(\alpha-1)^5 t^9}{1620} - \frac{5\alpha(\alpha-1)^6 t^8}{168} + \frac{(\alpha-1)^7 t^7}{63}\right)(sin^2 x + 3sinx + 1)^4.
\end{aligned}
$$

Then the approximate solution to (6.13) is given by $v(x,t) = v_0 + v_1 + v_2 + v_3$. We note that our proposed method the NDJM gives more terms in it's approximate solution than the NADM and the NHPM, we also observed this in the previous chapter.

In the Figures 6.1 to 6.4, the results from chapter 5 and the present chapter are compared.

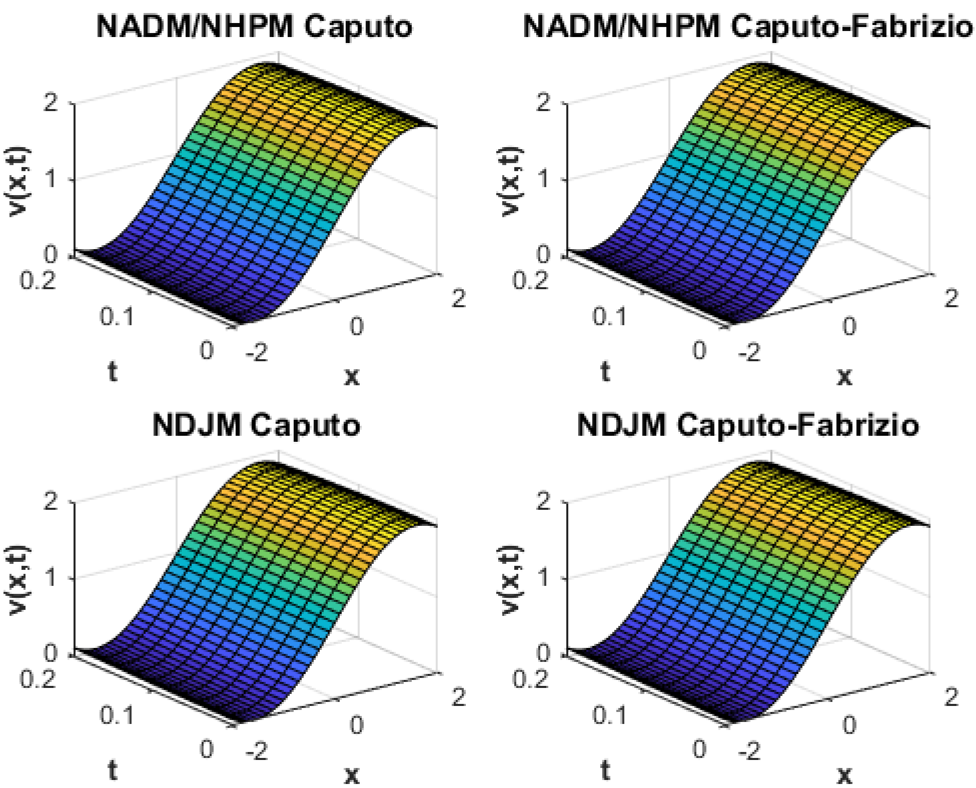

Figure 6.1: 3D plots of the approximate solution of the Fractional Klein-Gordon differential equation using the NTDM for $\mu = 2$

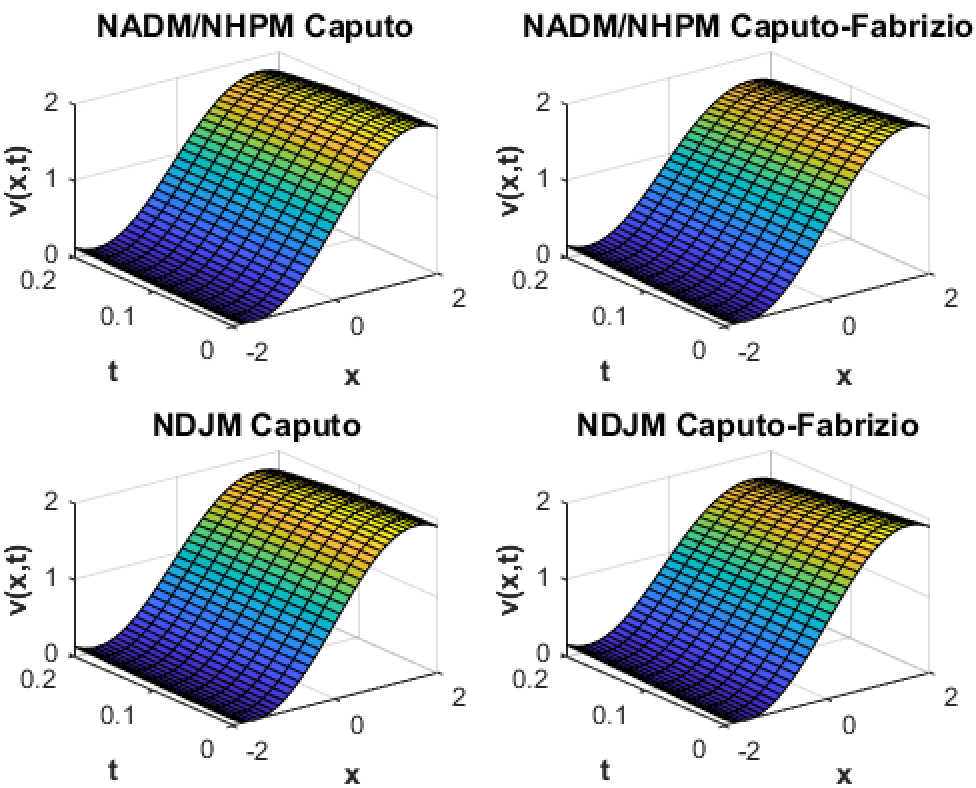

Figure 6.2: 3D plts of the approximate solution of the Fractional Klein-Gordon differential equation using the NTDM for $\mu = 1.7$

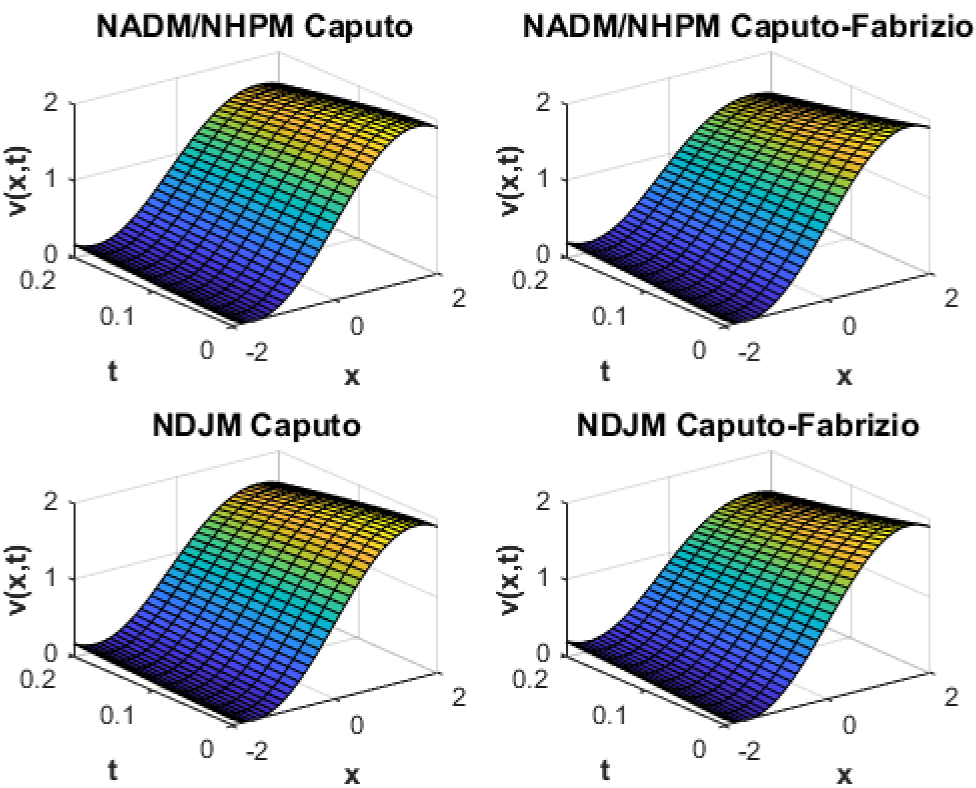

Figure 6.3: 3D plots of the approximate solution of the Fractional Klein-Gordon differential equation using the NTDM for $\mu = 1.4$

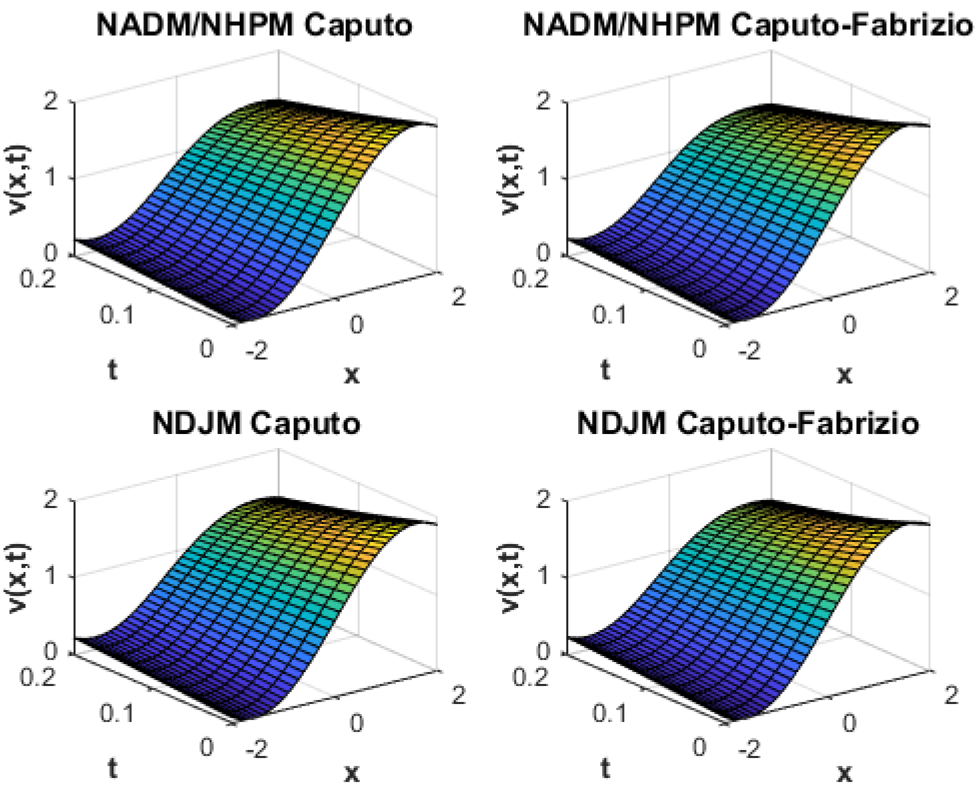

Figure 6.4: 3D plots of the approximate solution of the Fractional Klein-Gordon differential equation using the NTDM for $\mu = 1.1$

We note that from Figure 6.1 to 6.4, that there is a slight difference between the two fractional derivatives. In order to get a better insight, we do two dimensional plots from figure 6.5 to 6.8, we hold the value of x constant.

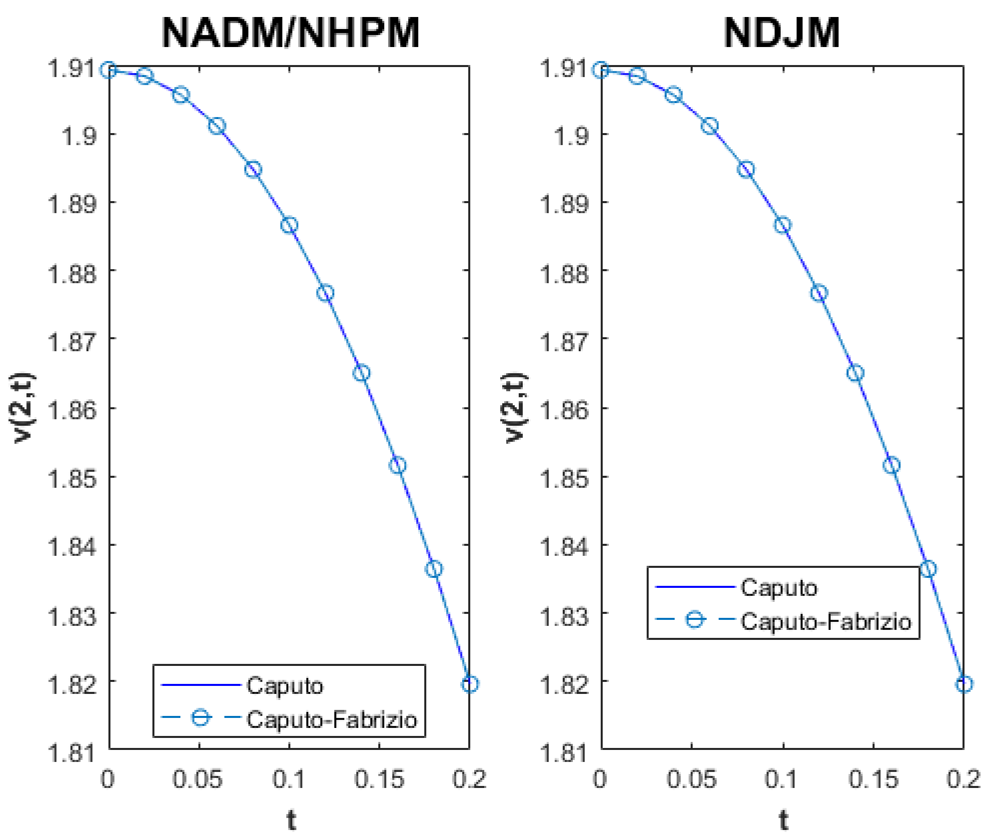

Figure 6.5: A plot of the approximate solution of the Fractional Klein-Gordon differential equation using the NTDM for $\mu = 2$ with $x = 2$

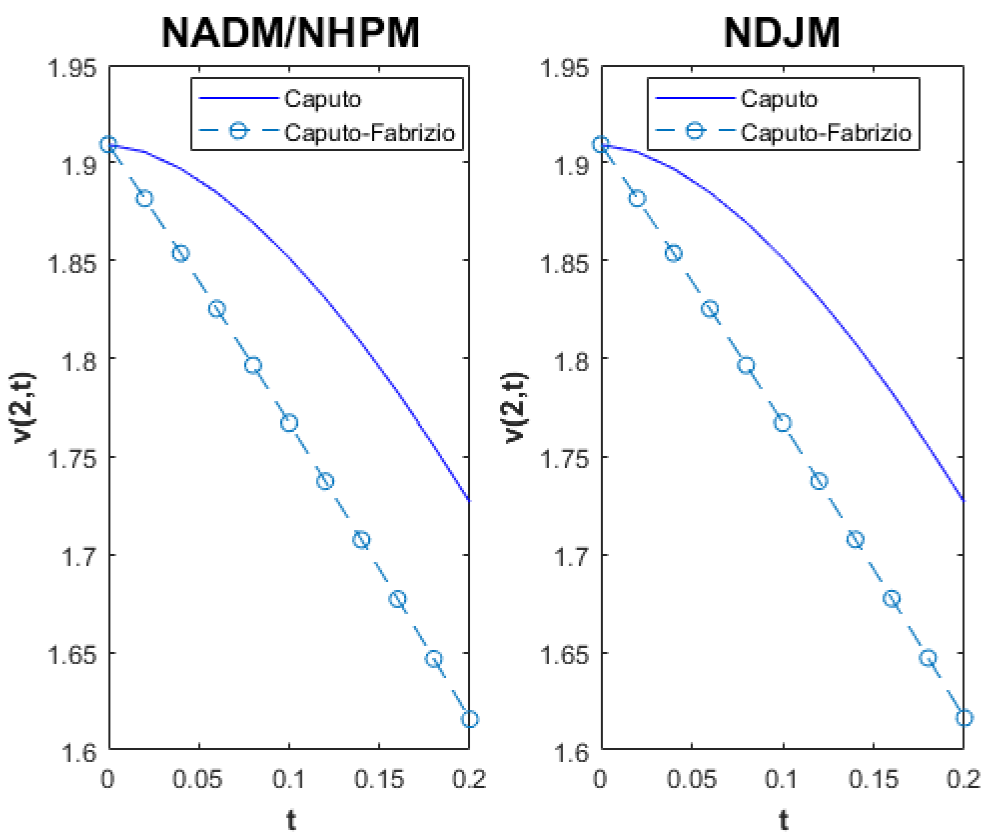

Figure 6.6: A plot of the approximate solution of the Fractional Klein-Gordon differential equation using the NTDM for $\mu = 1.7$ with $x = 2$

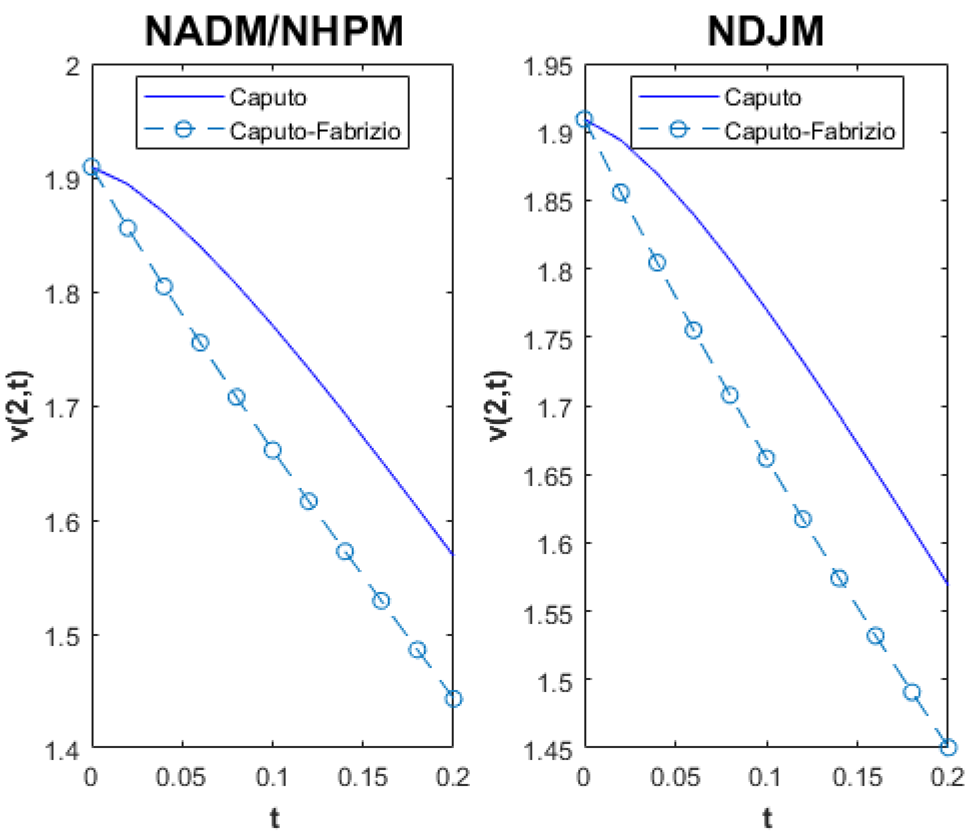

Figure 6.7: A plot of the approximate solution of the Fractional Klein-Gordon differential equation using the NTDM for $\mu = 1.4$ with $x = 2$

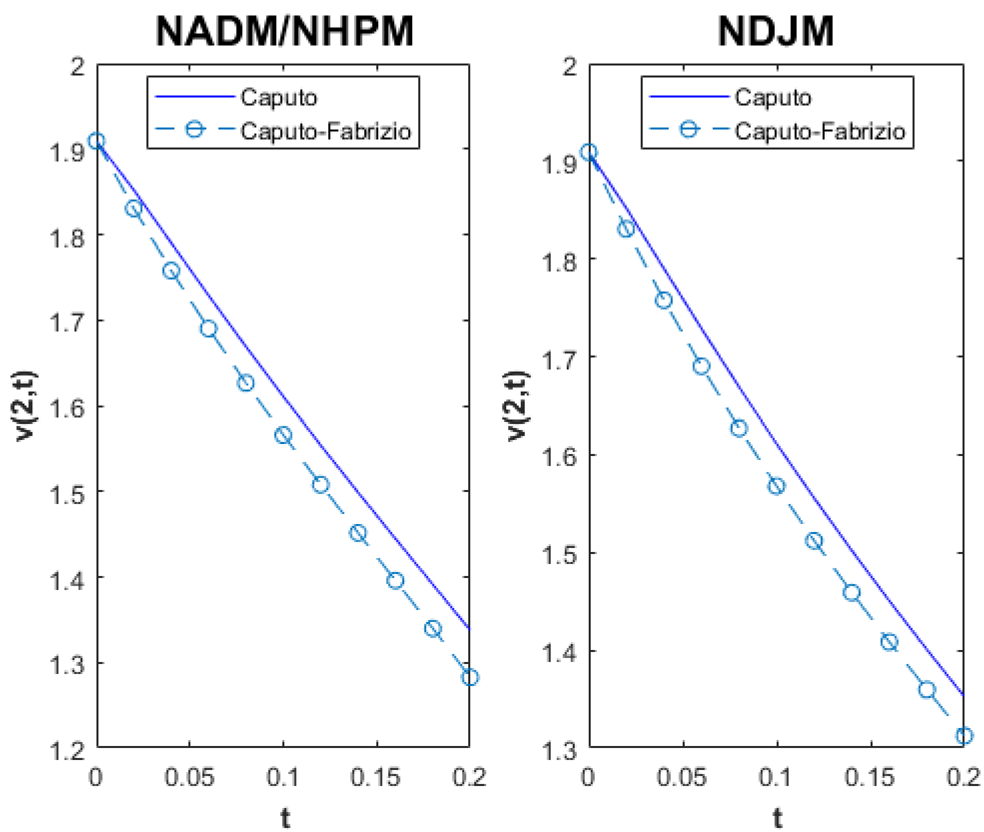

Figure 6.8: A plot of the approximate solution of the Fractional Klein-Gordon differential equation using the NTDM for $\mu = 1.1$ with $x = 2$

We notice in figure 6.5 with $\mu = 2$, the two fractional derivatives give the same results, this result is expected because at $\mu = 2$ the Fractional Klein-Gordon differential equation coincides with integer order Klein-Gordon differential equation.

Then as the value of μ is decreased from 2, from Figure 6.6 to 6.8, we note that the Caputo fractional derivative is affected more by the past as it shows slower changes, compared to the Caputo-Fabrizio fractional derivative which stabilises quickly, this is particularly evident in Figures 6.6 and 6.7.

Chapter 7

Summary

In Chapter 1, we introduced the dissertation with the necessary literature review and gave some background of the research. We also gave the important definitions and notations that were used throughout the dissertation.

In Chapter 2, we dealt with fractional calculus. We first gave the definitions of the special functions, these functions are the basis of fractional calculus. We gave the definition of the Riemann-Liouville's fractional integral and it's properties. Then we gave the formal definitions of the Riemann-Lioville, Caputo and Caputo-Fabrizio fractional derivatives. We also pointed out the connections between the three fractional derivatives.

In Chapter 3, we dealt with integral transforms, we mentioned various integral transforms. Our main focus was on the natural transform, we described the properties of this integral transform and gave examples of how it can be used to solve an ordinary differential equation. We made use of the relationship between the natural and Laplace transforms to derive the natural transforms of the Caputo and Caputo-Fabrizio fractional derivatives. We gave an example to illustrate the use of the natural transform to solve linear fractional ordinary differential equation. The main advantage of the integral transforms are to avoid the difficult integrations that may arise when solving differential equations, however they are incapable of handling non linearity.

In Chapter 4, we discussed the decomposition methods namely, Adomian decomposition method (ADM), homotopy perturbation method (HPM) and Daftardar-Jafari method

(DJM). We described the methodology of these decomposition methods and discussed their convergence and rates of convergence. We solved a non linear ordinary differential of fractional order using the decomposition methods, from the results, the ADM and HPM gave the same results that were different from the DJM. The DJM approximate solution produced extra terms in addition the ADM and HPM ones. We then did convergence analysis from the results of the example that we solved, the DJM proved to have a faster rate of convergence than the ADM and HPM, we concluded that this faster rate of convergence was attributed to the extra terms that the DJM produced. Although the decomposition methods have the capacity to handle non linearity, the major disadvantage is that most often they are difficult integrations that are encountered.

In Chapter 5, we combined the natural transform and the decomposition methods to come up with one method namely, the natural transform decomposition method (NTDM). This is a truly hybrid method that takes the advantages at the same time eliminating the disadvantages from both the natural and decomposition methods. This method avoids the difficult integrations that may arise when solving the differential equations by making use of the natural transform, and at the same time handling non linear differential equations by using the decomposition methods. We were able to apply the NTDM to a non linear Klein-Gordon differential equation of fractional order and compared our results with the Fractional reduced transform method. We suggested our own form of the NTDM, by combining the natural transform and the Daftardar-Jafari method (DJM), we had to show that the suggested technique had a faster rate of convergence than the Natural Adomian decomposition method (NADM) and the Natural homotopy perturbation method (NHPM).

In Chapter 6, we applied the NTDM to the same non linear Fractional Klein-Gordon differential equation as in Chapter 5, but we used the Caputo-Fabrizio fractional derivative. Our main aim of this chapter was to compare the Caputo and Caputo-Fabrizio fractional derivatives. We noticed from our results that the Caputo fractional derivative exhibits a slower response, it changes slowly with time, thus it is affected more by the past, the Caputo-Fabrizio shows a faster change with time and stabilises quickly. However, we have to point out that this difference doesn't mean one fractional derivative has the advantage over the other, it all depends on the situation that is to be mathematically modelled.

There are many areas that may be of future research that we encountered throughout the dissertation. However, we would point out two of them that we think are of great interest. The first one, the natural transform decomposition method can possibly be applied to fractional differential equations using the Atangana-Baleanu-Liouville fractional derivative, this is a new derivative that was soon developed after the Caputo-Fabrizio fractional derivative. The second one, the Mellin transform is one of the integral transforms that we mentioned, one might explore the possibility of combining the Mellin transform with the decomposition methods.

Appendix A

MATHEMATICA CODE

A.1 Mathematica code for Adomian Polynomials

```
u[x_] := x^2;
V = u[Sum[x_i λ^i, {i, 0, 10}]];
Do[Print["A", k, "=", 1/Factorial[k] D[V, {λ, k}] /. λ → 0], {k, 0, 6}]
```

$$u[x\_] := x^2;$$

$$V = u\left[\sum_{i=0}^{10} x_i \lambda^i\right];$$

$$\text{Do}\left[\text{Print}\left[\text{"A"}, k, \text{"="}, \frac{1}{\text{Factorial}[k]} D[V, \{\lambda, k\}] \text{ /. } \lambda \to 0\right], \{k, 0, 6\}\right]$$

A0= x_0^2

A1= $2\, x_0\, x_1$

A2= $\frac{1}{2}\,(2\, x_1^2 + 4\, x_0\, x_2)$

A3= $\frac{1}{6}\,(12\, x_1\, x_2 + 12\, x_0\, x_3)$

A4= $\frac{1}{24}\,(24\, x_2^2 + 48\, x_1\, x_3 + 48\, x_0\, x_4)$

A5= $\frac{1}{120}\,(240\, x_2\, x_3 + 240\, x_1\, x_4 + 240\, x_0\, x_5)$

A6= $\frac{1}{720}\,(720\, x_3^2 + 1440\, x_2\, x_4 + 1440\, x_1\, x_5 + 1440\, x_0\, x_6)$

Appendix B

Norms and Identities

B.1 Formulae for Norms and Identities

Let v_1 and $v_2 \in \mathbb{R}^n$. Then a function $\| \, . \, \|: \mathbb{R}^n \to \mathbb{R}$ is defined as a vector norm if the following properties are satisfied [10],

(i) $\| v_1 \| \geq 0$,

(ii) $\| v_1 \| = 0, \quad$ if and only if $v_1 = 0$,

(iii) $\| cv_1 \| = | c | \| v_1 \|, \quad c \in \mathbb{R}$

(iv) $\| v_1 + v_2 \| \leq \| v_1 \| + \| v_2 \|$,

property *iv* is called the triangle inequality.

The relation between the complex exponentials and trigonometric functions is expressed in the form

$$\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2}.$$
$$\sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i}.$$

Bibliography

[1] Abdel-Rady A. S, Rida S Z, Arafa A. A. M, Abedl-Rahim H. R, (2015). *Natural Transform for solving Fractional models*, J. Appl. Math. Phy, 3:1633-1644.

[2] Abuteen E, Fraihat A, Al-Smadi M, Khalil H and Khan R. A, (2016). *Approximate series solution of nonlinear Fractional Klein-Gordon equations using Fractional reduced differential transform method*, J. Math. Stat, 12:23-33.

[3] Adomian G, 1988. *A review of the decomposition method in applied mathematics*: J.Math. Anal, Appl, 135: 501-544

[4] Adomian G, 1989. *Nonlinear Stochastic Systems Theory and Applications to Physics*: Kluwer Academic.

[5] Adomian G and Race R, (1992). Noise term in decomposition solution series: Comput. Math. Appl., 24: 61-64.

[6] Al-Salti N, Karimov E, Sadarangani K, (2016). *On a Differential Equation with Caputo-Fabrizio Fractional Derivative of order* $1 < \beta \leq 2$ *and Application to Mass-Spring-Damper System*, Progr. Fract. Differ. Appl. 2, No. 4, 257-263.

[7] Babolian E, Azizi A, Saeidian J, (2009). *Some notes on using the Homotopy Perturbation Method for solving time dependent differential equations*, Math. Compt. Mode, 50:213-224.

[8] Balgacem F. B. M and Karaballi A. A, (2006). *Sumudu transform fundamental properties investigations and applications.*, J. Appl. Math and Stoch. Anal, 2006:1-23.

[9] Balgacem F. B. M and Silambarasan R, (2012). *Maxwell's equations by means of the natural transform*, Math. Eng. Sc. Aero, 3:313-323.

[10] Braess D, (2007) *Finite Elements, theory, fast solvers and applications in elasticity theory*, Cambridge University Press.

[11] Bhalekar S and Patade J, (2016). *An Analytical Solution of Fisher's Equation Using Decomposition Method*, J. Comp. Appl. Math , 6:123-127

[12] Caputo M and Fabrizio M, (2015). *A new definition of Fractional Derivative without singular kernel*, Progr. Fract. Differ. Appl, 2:73-85

[13] Daftardar-Gejji V and Bhalekar S, (2008). *Solving fractional diffusion-wave equations using a new iterative method*, J. Theo. Appl, 11:193-202

[14] Daftardar-Gejji V and Jafari H, (2006). *An iterative method for solving non linear functional equations*, J. Math. Anal. Appl. 316:753-763.

[15] Doungmo Goufo E. F, (2016). *Application of the Caputo-Fabrizio Fractional Derivative without Singular Kernel to Korteweg-de Vries-Bergers Equation*, Math. Model. Ana 21(2): 188-198.

[16] Elgazery N. S, (2008). *Numerical solution for the Falkner-Skan equation*, Chaos Soliton and Fractals, 35: 738-746.

[17] Haberman R, (2004) *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, PEARSON Education.

[18] Hemeda A. A, (2012). *Homotopy perturbation method for solving systems of non linear coupled equations*, Appl. Math. Sc, 6:4787-4800.

[19] Hosseini M. M and Nasabzadeh H, (2006). *On the Convergence of Adomian decomposition method*, Appl. Math. Computation, 182:536-543.

[20] Hussain M and Khan M, (2010). *Modified Laplace decomposition method*: Appl. Math. Sci., 36(3): 1769-1783.

[21] Jafari H and Daftardar-Gejji V, (2006). *Solving a system of non linear fractional differential equations using Adomian decomposition*, J. Comp. Appl. Math, 196:644-651.

[22] Jafari H, Khalique C. M and Nazari M, (2011). *Application of the Laplace Decomposition Method for solving Linear and nonlinear fractional diffusion wave equations*, Appl. Math. Letters 24:1799-1805.

[23] Jafari H and Ncube M. N, (2018). *Fourier-Natural transform method for solving a class of fractional partial differential equations*, Far East Journal of Math. Sci (Accepted).

[24] Jafari H, Ncube M. N, Moshokoa S. P, Makhubela L, (2018). *Natural Daftardar-Jafari method for solving fractional partial differential equations*, Nonlinear Dynamics and Systems Theory (Accepted).

[25] Johnston S. J, Jafari H, Moshokoa S. P, Ariyan V. M and Baleanu D, (2016). *Laplace Homotopy Pertubation Method for Burgers equation with space and time fractional order*, Open Phys:14:247-252.

[26] Kazem S, (2013) *Exact Solution of Some Linear Fractional Differential Equations by Laplace Transform*, J.Nonlinear. Sc, 16:3-11

[27] Khan M and Gondal M. A, (2010). *A new analytical solution of foam drainage equation by Laplace decomposition method*: J. Advance Research Diff.Eq., 2(3): 53-64.

[28] Khan Z. H and Khan W, (2008). *N-Transform Properties and Applications*, NUST Journal Engineering Sciences, 1:127-133.

[29] Khuri S. A, (2001). *A Laplace decomposition algorithm applied to a class of nonlinear differential equations*: J. Appl. Math., 4: 141-145.

[30] Khuri S. A, (2004). *A new approach to Bratu's problem*: Appl. Math. Comput., 147: 131-136.

[31] Kiymaz O, (2009). *An algorithm for solving Initial Value Problems using Laplace decomposition method* : Appl. Math. Sci., 3: 1453-1459.

[32] Lou X. G, (2005). *A two-step Adomian decomposition method*: Appl. Math. Comput., 170: 570-583.

[33] Maitama S, (2014). *A new approach to Linear and Nonlinear Schrodinger Equations using the Natural Decomposition Method*, International Mathematical Forum, 9:835-847.

[34] Maitama S and Abdullahi I, (2016). *A New Analytical Method for Solving Linear and Nonlinear Fractional Partial Differential Equations*, Progr. Fract. Differ. 2(4):247-256

[35] Morales-Delgado V. F, Gómez-Aguilar J. F, Yépez-Martínez H, Baleanu D, Escobar-Jimenez R. F and Olivares-Peregrino V. H, (2016). *Laplace homotopy analysis method for solving linear partial differential equations using a fractional derivative with and without a singular kernel*, Advances in Difference Equations 2016:164.

[36] Podlubny I, (1999). *An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their solution and some of their applicatons*, Academic Press.

[37] Rawashdeh M. S and Maitama S, (2014). *Solving coupled system of Nonlinear Partial Differential Equations using the Natural Decomposition Method*, J. Pure. Appl. Math, 92(5):757-776.

[38] Saberi-Nadjafi J and Ghorbani A, (2009). *He's homotopy perturbation method:An effective tool for solving integral and integro-differential equations*, Comp. Math. Appl, 58:2379-2390.

[39] Saha Ray S, (2014). *New Approach for General Convergence of the Adomian Decomposition Method*, J. Appl. Sc, 32(11):2264-2268.

[40] Sheikh N. A, Ali F, Saqib M, Khan I, Jan S. A. A, Alshomrani A. S and Alghamdi M. S, (2017). *Comparison and analysis of the Atangana-Baleanu and Caputo-Fabrizio*

fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction, Results in Physics 7:789-800.

[41] Song Y and Kim H, 2014. *The solution of Volterra Integral equation of the Second Kind by using the Elzaki Transform*, Appl. Math. Sc, 8(11):525-530.

[42] Tatari M, Dehghan M and Razzaghi M, (2007). *Application of the Adomian decomposition method for the Fokker-Planck equation*, Math. Comp. Modelling 45:639-650.